Nonstationary models with long memory

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1 Introduction

Long memory processes have been used for some decades to model data that show a strong persistence of their correlations. Such data can typically be found in the applied sciences such as hydrology, geophysics, climatology and telecommunication (e.g. teletraffic data) but also in economics and in finance, e.g. for modelling (realized) volatility of exchange rate data or stocks. This phenomenon is often referred to as *long range dependence*. More precisely, one says that a second order stationary process X_t is *long range dependent* if its covariance function γ satisfies

$$\sum_{t\in\mathbb{Z}} |\gamma(t)| = \infty \tag{1}$$

Otherwise, if moreover

$$\sum_{t\in\mathbb{Z}} \gamma(t) > 0\,, \tag{2}$$

one says that (X_t) is short range dependent. In the last case where

$$\sum_{t\in\mathbb{Z}} \gamma(t) = 0\,, \tag{3}$$

one says that (X_t) is *long-range dependent with negative correlations*. These are the most widely used (and most simple) definitions of long, short, or negative long range dependence. However it may appear as being too restrictive, in the first place because it only applies to processes with finite variance. Many insights about alternative approaches for defining long range dependence can be found in Samorodnitsky [2006]. One of them is to view long range dependent processes as a class of stationary models approaching the border of non-stationarity, resulting in a phenomenon that may be seen as a *phase transition* from the i.i.d. case to the non-stationary case. Things can be made more precise by introducing a *long memory parameter*. Again several ways are possible, the most simple is probably to define d by imposing that, as $n \to \infty$,

$$\mathrm{var}\left(\sum_{k=1}^{n} X_k\right) \sim c\, n^{1+2d}\,, \tag{4}$$

where c is a positive constant. (A more general approach consists in replacing c by a function $L(n)$ slowly varying as $n \to \infty$; we adopt the more restrictive condition $L(n) \sim c$ here for simplicity). In the case of short-range dependence as defined by (2), Eq (4) holds with $d = 0$. Long-range dependence corresponds to $d > 0$ and negative long-range dependence to $d < 0$, although the definitions above do not imply the asymptotic equivalence (4). We see that for a weakly stationary process (X_t), d in (4) cannot be larger than $1/2$. Hence, long-range dependence indeed corresponds to some transition, $0 < d < 1/2$, between short-range dependence, $d = 0$, and non-stationarity, say $d \geq 1/2$. The literature on stationary long-range dependent processes is huge (see e.g. the references in the recent

survey paper Faÿ et al. [2009]). Such processes received much attention in financial time series since they were introduced by Granger and Joyeux [1980] in this context.

In this contribution, we examine some cases where long memory models cross the border of non-stationarity. A well known case is the extension of the long memory parameter d to values larger than $1/2$. We will recall the different approaches that have been proposed to do so for linear processes in Section 3. Before that we introduce generalized processes and generalized spectral measures in Section 2. The goal here is to promote these tools to define long memory models with stationary increments as they provide a natural extension to the case of weakly stationary processes. We conclude in Section 4 by mentioning a truly non-stationary (that is, neither stationary, nor increment stationary) model: the locally stationary long memory process.

2 Generalized processes

2.1 Generalized fractional Brownian motion

As can be seen from (4), the long memory parameter is related to a scaling behavior at large scales. Hence the concept of *self-similarity*, which imposes a scaling behavior at all scales, is a very first step for introducing long memory models. In the Gaussian case, one obtains the celebrated fractional Brownian motion (fBm).

Definition 2.1 (Fractional Brownian motion)**.** Let $H \in (0,1)$. The *Fractional Brownian motion (fBm)* with *Hurst index* H is a H-self-similar Gaussian process $\{B_t^{(H)},\, t \in \mathbb{R}\}$ with stationary increments.

It follows from this definition that $B_0^{(H)} = 0$ p.s., $\mathbb{E}[B_t^{(H)}] = 0$ and the variogram is given by

$$\text{(5)} \quad \operatorname{var}(B_t^{(H)} - B_s^{(H)}) = \sigma^2 |t-s|^{2H} \ .$$

Thus the covariance function reads

$$\operatorname{cov}(B_s^{(H)}, B_t^{(H)}) = \frac{\sigma^2}{2} \left\{ |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right\}, \quad s,t \in \mathbb{R} \ .$$

If $H = 1/2$, $B^{(H)}$ has independent increments and thus is the *Brownian motion*. Now for $H \in (0,1)\backslash\{1/2\}$, we may define $B^{(H)}$ as

$$B_t^{(H)} = \int \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right] \mathrm{d}M_s, \quad t \in \mathbb{R} \ ,$$

where $\{M_s,\, s \in \mathbb{R}\}$ denotes the Brownian motion. The fBm is only defined for $H \in (0,1)$ but can be extended to any arbitrary Hurst, parameter $H \in \mathbb{R}$ by using generalized processes as inYaglom [1958] or Major [1981]. The following definition is used in Moulines et al. [2007].

Definition 2.2. The *generalized fractional Brownian motion* $B^{(H)}$, where $H \in \mathbb{R}$ is parameterized by a family $\Theta_{(d)}$ of "test" functions θ defined on $\mathbb{R}$ and is defined as follows: $\{B^{(H)}(\theta),\, \theta \in \Theta_{(d)}\}$ is a mean zero Gaussian process with covariance

$$\text{(6)} \quad \operatorname{cov}\left(B^{(H)}(\theta_1),\, B^{(H)}(\theta_2) \right) = \int_{\mathbb{R}} |\xi|^{-2d}\, \theta_1^*(\xi)\, \overline{\theta_2^*(\xi)}\, \mathrm{d}\xi \ ,$$

where $d = H + 1/2$, θ^* is the Fourier transform of θ,

$$\theta^*(\xi) = \int \theta(t)\, \mathrm{e}^{-\mathrm{i}\xi t} \mathrm{d}t \ ,$$

and $\Theta_{(d)}$ is a set of test functions θ satisfying

$$\int_{\mathbb{R}} |\xi|^{-2d}\, |\theta^*(\xi)|^2\, \mathrm{d}\xi < \infty \ .$$

In Major [1981], the test function θ is any function of the Schwartz space. Here the restriction to a smaller set $\Theta_{(d)}$ allows to take $d \geq 1/2$.

Let us explain why this is an extension to the standard fBm indexed by the continuous time when $0 < H < 1$. Clearly, $\Theta_{(d)}$ can be taken exactly as the class of tempered distributions θ such that

$$\int_{\mathbb{R}} |\xi|^{-2d} \, |\theta^*(\xi)|^2 \, \mathrm{d}\xi < \infty \, .$$

Let $0 < H < 1$ i.e. $1/2 < d < 3/2$. Then, denoting by δ_x the Dirac distribution with support $\{x\}$, $\delta_t - \delta_0 \in \Theta_{(d)}$ for all $t \in \mathbb{R}$. Moreover the process $\{B^{(H)}(\delta_t - \delta_0),\, t \in \mathbb{R}\}$ is a H-self-similar Gaussian process, with stationary increments; thus it is the fBm $\{B_t^{(H)},\, t \in \mathbb{R}\}$, up to a multiplicative constant.

2.2 Generalized spectral measure

Following an approach similar to that of Section 2.1, one may extend discrete time indexed processes by sequence indexed processes. Although it may appear unnecessarily abstract at first sight, this extension shall be quite efficient for providing a simple spectral description of second order properties of increment stationary processes.

Given a process $X = \{X_k,\, k \in \mathbb{Z}\}$, we may define a *generalized process* indexed by sequences $\mathrm{h} = (\mathrm{h}_k) \in \mathbb{C}^{\mathbb{Z}}$ as follows:

$$(7) \qquad X(\mathrm{h}) = \sum_{k \in \mathbb{Z}} \mathrm{h}_k X_k \, .$$

This approach allows to define a spectral measure in a weaker context than the usual covariance stationarity assumption.

Definition 2.3 (Generalized spectral measure)**.** Let ν be a (non-necessarily finite) non-negative measure on the Torus $\mathbb{T} = [-\pi, \pi)$ and V be a linear space of finitely supported (f.s.) sequences in $\mathbb{C}^{\mathbb{Z}}$. The process $X = \{X_k,\, k \in \mathbb{Z}\}$ is said to admit generalized spectral measure ν on V if for any $\mathrm{h} \in V$, $X(\mathrm{h})$ has finite variance given by

$$(8) \qquad \mathrm{var}(X(\mathrm{h})) = \int_{\mathbb{T}} |\mathrm{h}^*(\lambda)|^2 \; \mathrm{d}\nu(\lambda) \, ,$$

where

$$(9) \qquad \mathrm{h}^*(\lambda) = \sum_{k \in \mathbb{Z}} \mathrm{h}_k \mathrm{e}^{-\mathrm{i}k\lambda} \, .$$

Of course it follows form (8) and the fact that $\mathrm{h} \mapsto X(\mathrm{h})$ is linear that, for all $\mathrm{h}^{(1)}$ and $\mathrm{h}^{(2)} \in V$,

$$\mathrm{cov}(X(\mathrm{h}^{(1)}), X(\mathrm{h}^{(2)})) = \int \mathrm{h}^{(1)*}(\lambda) \overline{\mathrm{h}^{(2)*}(\lambda)} \, \mathrm{d}\nu(\lambda) \, .$$

If ν is a finite measure on the Torus, and V contains all f.s. sequences (in particular, for each $t \in \mathbb{Z}$, there is $\mathrm{h} \in V$ with $X(\mathrm{h}) = X_t$), then X is covariance stationary and ν is its usual spectral measure. Conversely if X is covariance stationary and ν is its spectral measure, then (8) holds for all $\mathrm{h} \in V_0$, where V_0 denotes the set of all f.s. sequences.

Another case of interest is studied in the following section. Before that, let us observe that one may proceed similarly in continuous time. For instance, in view of (6), the generalized fBm defined in Section 2.1 admits the generalized spectral density $|\lambda|^{-2d}$ on $\lambda \in \mathbb{R}$ (instead of the Torus in the discrete time case).

2.3 Increment stationary processes

Stationarity of the increments is commonly assumed in time-series analysis. A celebrated example in econometrics is the class of ARIMA(p,k,q) processes, for which $\mathbf{\Delta}^k Y$ is a stationary ARMA(p,q) process.

Definition 2.4 (Increment stationary processes)**.** Let k be a non-negative integer. We say that a process $X = \{X_t,\, t \in \mathbb{Z}\}$ is a k-th order increment (covariance) stationary processes if $\mathbf{\Delta}^k X$ is (covariance) stationary, where $(\mathbf{\Delta} X)_t = X_t - X_{t-1}$ and $\mathbf{\Delta}^k$ is defined recursively, $\mathbf{\Delta}^{k+1} = \mathbf{\Delta} \circ \mathbf{\Delta}^k$.

For any $d \in \mathbb{R}$, define V_d as the set of all f.s. sequences $(\mathrm{h}_k) \in \mathbb{C}^{\mathbb{Z}}$ such that

$$\int_{-\pi}^{\pi} |\mathrm{h}^*(\lambda)|^2 \; |\lambda|^{-2d} \mathrm{d}\lambda < \infty \; . \tag{10}$$

The definition of V_d is of interest mainly for $d > 1/2$, since otherwise $V_d = V_0$. Then k-th order increment covariance stationary processes can entirely be described by processes admitting a generalized spectral measure on V_k. To see why, we first derive the following lemma.

Lemma 2.5. *Let k be a non-negative integer. Then, for all $d \in [k-1/2, k+1/2)$, V_d is the set of f.s. sequences $\mathrm{h} \in \mathbb{C}^{\mathbb{Z}}$ such that there exists a f.s. sequences $\tilde{\mathrm{h}} \in \mathbb{C}^{\mathbb{Z}}$ such that $\mathrm{h} = \mathbf{\Delta}^k \tilde{\mathrm{h}}$.*

Proof. Suppose that $\mathrm{h} = \mathbf{\Delta}^k \tilde{\mathrm{h}}$ with $\tilde{\mathrm{h}}$ f.s. Then $\tilde{\mathrm{h}}^*$ is bounded and

$$|\mathrm{h}^*(\lambda)|^2 = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2k} \; |\tilde{\mathrm{h}}^*(\lambda)|^2 \; .$$

It follows that $\mathrm{h} \in V_d$ for all $d < k + 1/2$.

We prove the converse result by induction on $k \geq 0$, namely, if $\mathrm{h} \in V_d$ for some $d \geq k - 1/2$, then there is a f.s. sequence $\tilde{\mathrm{h}}$ such that $\mathrm{h} = \mathbf{\Delta}^k \tilde{\mathrm{h}}$. It is obviously true for $k = 0$. Suppose that it is true for some $k \geq 0$. Let $\mathrm{h} \in V_d$ for some $d \geq k + 1/2$. Observe that $|\mathrm{h}^*|^2$ is a trigonometric polynomial, say, for some $n \in \mathbb{Z}$ and $p \geq 0$,

$$|\mathrm{h}^*(\lambda)|^2 = \left| \sum_{j=0}^{p} \mathrm{h}_{j+n} \mathrm{e}^{-\mathrm{i}j\lambda} \right|^2 \; .$$

Then, for the integral in (10) to be finite, it is necessary that the polynomial $z \mapsto \sum_{j=0}^{p} \mathrm{h}_{j+n} z^j$ vanishes at $z = 1$. Hence factorizing $1 - z$ in this polynomial, we may write h as $\mathbf{\Delta}\check{\mathrm{h}}$ for some f.s. $\check{\mathrm{h}}$. Moreover,

$$|\mathrm{h}^*(\lambda)|^2 = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^2 \; |\check{\mathrm{h}}^*(\lambda)|^2 \; .$$

Thus, $\mathrm{h} \in V_d$ implies $\check{\mathrm{h}} \in V_{d-1}$. Applying the induction hypothesis, we get that $\mathrm{h} = \mathbf{\Delta}^{k+1} \tilde{\mathrm{h}}$ for some f.s. sequence $\tilde{\mathrm{h}}$. □

Observe that when $\mathrm{h} = \mathbf{\Delta}^k \tilde{\mathrm{h}}$ holds for two f.s. sequences, $\tilde{\mathrm{h}}$ is uniquely defined by

$$\tilde{\mathrm{h}} = \mathbf{\Delta}^{-k} \mathrm{h} \; .$$

where $\mathbf{\Delta}^{-k}$ is defined iteratively by

$$(\mathbf{\Delta}^{-1} \mathrm{h})_t = \sum_{s=-\infty}^{t} \mathrm{h}_s \; ,$$

and, for $k \geq 1$, $\mathbf{\Delta}^{-k-1} = \mathbf{\Delta}^{-1} \circ \mathbf{\Delta}^{-k}$. We now characterize increment covariance stationary processes as processes having a generalized spectral measure on a space V_d.

Theorem 2.6. *Let k be a non-negative integer. A process $X = \{X_t,\, t \in \mathbb{Z}\}$ is k-th order increment covariance stationary with the spectral measure of $\mathbf{\Delta}^k X$ having mass 0 at the origin if and only if X admits a generalized spectral measure ν on V_d for some $d \in [k - 1/2, k + 1/2)$. Moreover, $\mathbf{\Delta}^k X$ has spectral measure ν_k given by*

$$\mathrm{d}\nu_k(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2k}\, \mathrm{d}\nu(\lambda)x \; . \tag{11}$$

Proof. Let X be k-th order increment covariance stationary and denote by ν_k the spectral measure of $\mathbf{\Delta}^k X$. Assume that ν_k has mass 0 at the origin, $\nu_k(\{0\}) = 0$. Let ν be the measure defined on the Torus by

$$\mathrm{d}\nu(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2k}\, \mathrm{d}\nu_k(\lambda) \; .$$

Take now $\mathrm{h} \in V_d$ with $d \in [k - 1/2, k + 1/2)$. By Lemma 2.5, we may write $\mathrm{h} = \mathbf{\Delta}^k \tilde{\mathrm{h}}$ with $\tilde{\mathrm{h}}$ denoting a f.s. sequence. It follows that

$$\begin{aligned} X(\mathrm{h}) = [\mathbf{\Delta}^k X](\tilde{\mathrm{h}}) &= \int |\tilde{\mathrm{h}}^*(\lambda)|^2\, \mathrm{d}\nu_k(\lambda) \\ &= |\tilde{\mathrm{h}}^*(0)|^2 \nu_k(\{0\}) + \int |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2k} |\tilde{\mathrm{h}}^*(\lambda)|^2\, \mathrm{d}\nu(\lambda) \\ &= \int |\mathrm{h}^*(\lambda)|^2\, \mathrm{d}\nu(\lambda) \; , \end{aligned}$$

where we used that $\nu_k(\{0\}) = 0$.

Suppose now that X has generalized spectral measure ν on V_d with $d \in [k - 1/2, k + 1/2)$. Then for all f.s. sequence $\tilde{\mathrm{h}} \in \mathbb{C}^{\mathbb{Z}}$, $\mathbf{\Delta}^k \tilde{\mathrm{h}} \in V_d$ by Lemma 2.5 and

$$\mathrm{var}([\mathbf{\Delta}^k X](\tilde{\mathrm{h}})) = \mathrm{var}(X(\mathbf{\Delta}^k \tilde{\mathrm{h}})) = \int |\tilde{\mathrm{h}}^*(\lambda)|^2\, |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2k}\, \mathrm{d}\nu(\lambda) = \int |\tilde{\mathrm{h}}^*(\lambda)|^2\, \mathrm{d}\nu_k(\lambda) \; ,$$

where ν_k is defined by (11). This shows that $\mathbf{\Delta}^k X$ is covariance stationary with spectral measure ν_k. □

3 The long memory parameter

3.1 Increment stationary case

A quite general way to ensure the behavior (4) for a covariance stationary process X is based on the spectral measure of X as in the following definition. It appears that such an approach was first introduced in Adenstedt [1974].

Definition 3.1 (Long memory parameter, $M(d)$ process, $d < 1/2$)**.** We say that a covariance stationary process X has memory parameter d if its spectral measure ν is absolutely continuous in a neighborhood of the origin and, as $\lambda \to 0$,

$$\frac{\mathrm{d}\nu(\lambda)}{\mathrm{d}\lambda} \sim c|\lambda|^{-2d} \; ,$$

where c is a positive constant. The *short-range spectral measure* ν^* of X is then defined by

$$\mathrm{d}\nu^*(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2d} \mathrm{d}\nu(\lambda) \; .$$

To allow $d \geq 1/2$, we now extend the definition of long memory to increment stationary processes.

Definition 3.2 (Long memory parameter, $M(d)$ process, $d \in \mathbb{R}$)**.** The process X is said to have memory parameter $d \in \mathbb{R}$ and short-range spectral measure ν^* if for any non-negative integer $k > d - 1/2$, its k-th order difference $\mathbf{\Delta}^k X$ is covariance stationary with memory parameter $d - k$ and short-range spectral measure ν^*. We will equivalently say that X is an $M(d)$ process.

From Theorem 2.6, we obtain that the previous definition can be equivalently stated using generalized spectral measures.

Corollary 3.3. *A process X is an $M(d)$ process, $d \in \mathbb{R}$, if and only if it admits a generalized spectral measure ν on V_d that is absolutely continuous in a neighborhood of the origin and such that, as $\lambda \to 0$,*

$$\frac{\mathrm{d}\nu(\lambda)}{\mathrm{d}\lambda} \sim c|\lambda|^{-2d} ,$$

where c is a positive constant. The short-range spectral measure ν^* *of X is given by*

$$\mathrm{d}\nu^*(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{2d}\mathrm{d}\nu(\lambda) .$$

Standard examples of $M(d)$ processes can be found in Faÿ et al. [2009]. The fBm in discrete time $\{B_t^{(H)}, t \in \mathbb{Z}\}$ is one of them, with $d = H + 1/2$, see Definition 2.1. We may also mention its increment process, $\mathbf{\Delta}B^{(H)}$, called the fractional Gaussian noise (fGn), which is an $M(H - 1/2)$ process. However, the most widespread models are obtained using the *fractional integration operator*, as it is the case for the class of FARIMA models. We give some insights to this in the next section.

3.2 Fractional integration and long memory

The extension of the definition of long memory parameter to $d \geq 1/2$ based on k-th order increment stationarity was proposed by Hurvich and Ray [1995]. An alternative way to introduce a long memory parameter $d \geq 1/2$ originates from Robinson [1994]. It is based on the fractional integration operator $\mathbf{\Delta}^{-a}$ and by imposing that X_t starts afresh at time zero, which can be written compactly as

$$[\mathbf{\Delta}^d X]_t = \epsilon_t \mathbb{1}_{t\geq 1}, \quad t \in \mathbb{Z} ,$$

where $\{\epsilon_t, t \in \mathbb{Z}\}$ is a short memory process, say a weak white noise, and, for any $a \in \mathbb{R}$ and any sequence (x_t) with supports bounded away from $-\infty$,

$$[\mathbf{\Delta}^{-a} x]_t = \sum_{k\geq 0} \frac{a(a+1)\dots(a+k-1)}{k!} x_{t-k}$$

The statistical inference for this model is studied in Shimotsu and Phillips [2005], where the following solution to the above equation is used

$$X_t = \sum_{k=0}^{t-1} \frac{d(d+1)\dots(d+k-1)}{k!} \epsilon_{t-k}, \ t \geq 1 . \tag{12}$$

Such processes will be referred to as *truncated fractional integrated processes*. A drawback of this model is that it is not stationary, even for $d < 1/2$, and cannot be made so by the simple addition of an initial condition,

$$X_t = X_0 + \sum_{k=0}^{t-1} \frac{d(d+1)\dots(d+k-1)}{k!} \epsilon_{t-k}, \ t \geq 1 ,$$

except in the very special case where d is non-positive integer.

A more natural way to extend the fractional integration operator to the non-stationary case is to rely on increment stationarity, or equivalently, as shown in Theorem 2.6, to use generalized processes. Indeed, suppose that $Y = \{Y_t, t \in \mathbb{Z}\}$ is a centered process and admits a generalized spectral measure ν on V_d for some $d \in \mathbb{R}$. Define by Y^* the corresponding spectral representation process, that is the process indexed on

$$V_d^* = \{\mathrm{h}^*, \mathrm{h} \in V_d\} ,$$

and defined by

$$Y^*(\mathrm{h}^*) = Y(\mathrm{h}),\ \mathrm{h} \in V_d\ .$$

Then, using that $g \mapsto Y^*(g)$ is an isometric linear operator from V_d^*, endowed with the inner product

$$(f,g)_\nu = \int_{\mathbb{T}} h\overline{g}\,\mathrm{d}\nu\ ,$$

to the set of L^2 random variables, it can be extended to the closure of V_d^* with respect to the $L^2(\nu)$ norm. The following lemma shows that it is the whole space $L^2(\nu)$.

Lemma 3.4. *Let $d \in \mathbb{R}$ and ν be a generalized spectral measure on V_d. If $d \geq 1/2$ assume that $\nu(\{0\}) = 0$. The closure of V_d^* in $L^2(\nu)$ is $L^2(\nu)$.*

Proof. Let k be the integer such that $d \in [k-1/2, k+1/2)$. If $k \leq 0$ then $V_d = V_0$ and ν is a finite measure on the Torus. If $k \geq 1$, the function Δ^k, where $\Delta(\lambda) = 1 - \mathrm{e}^{-\mathrm{i}\lambda}$, belongs to V_d^* and, by Lemma 2.5, the measure $\tilde{\nu}$ defined on the Torus by

$$\mathrm{d}\tilde{\nu}(\lambda) = |\Delta(\lambda)|^{2k}\mathrm{d}\nu(\lambda)$$

is a finite measure. Hence V_0 is a dense subset of $L^2(\tilde{\nu})$. Now, by Lemma 2.5, we have

$$V_d^* = \left\{\Delta^k \times g,\ g \in V_0^*\right\}\ ,$$

and, using the assumption $\nu(\{0\}) = 0$, we easily see that

$$L^2(\nu) = \left\{\Delta^k \times g,\ g \in L^2(\tilde{\nu})\right\}\ ,$$

and, moreover, there exists $C > 0$ such that for all $g \in L^2(\tilde{\nu})$,

$$C\,\|g\|_{L^2(\tilde{\nu})} \leq \|\Delta^k \times g\|_{L^2(\nu)} \leq C^{-1}\,\|g\|_{L^2(\tilde{\nu})}\ .$$

The conclusion of the lemma follows. □

Observe that the assumption $\nu(\{0\}) = 0$ can be made without loss of generality since for $d \geq 1/2$ the right-hand side of (8) *does not depend* on $\nu(\{0\})$ for any $\mathrm{h} \in V_d$. This is again a consequence of Lemma 2.5. Moreover it allows to uniquely define the generalized spectral measure on V_d. Hence this assumption can be included in the definition of the generalized spectral measure of increment covariance stationary processes. We will say that the generalized spectral measure ν satisfying this condition is the *reduced* generalized spectral measure. We can now define the spectral representation as a process defined on $L^2(\nu)$ which is a natural extension of the weakly stationary case.

Definition 3.5. Let $Y = \{Y_t,\ t \in \mathbb{Z}\}$ be a centered process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with reduced generalized spectral measure ν on V_d for some $d \in \mathbb{R}$. The spectral representation Y^* of Y is defined as the unique isometric linear operator from $L^2(\mathbb{T}, \mathbb{B}(\mathbb{T}), \nu)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that, for all $\mathrm{h} \in V_d$,

$$Y^*(\mathrm{h}^*) = Y(\mathrm{h})\ .$$

An interesting consequence of this description of increment stationary processes is the following result that allows to define a *fractional differencing operator* of arbitrary order $x \in \mathbb{R}$.

Definition 3.6 (Fractional differencing operator)**.** Let $a \in \mathbb{R}$ and let $X = \{X_t,\ t \in \mathbb{Z}\}$ be a centered process a with reduced generalized spectral measure ν on V_d for some $d \in \mathbb{R}$. The *a-th order fractionally integrated process* $Y = \{Y_t,\ t \in \mathbb{Z}\}$, denoted by $Y = \mathbf{\Delta}^{-a}X$ is defined as the generalized process defined on V_{d+a} by

$$Y(\mathrm{h}) = X^*(\Delta^{(-a)} \times \mathrm{h}^*),\ \mathrm{h} \in V_{d-a}\ ,$$

where $\Delta^{(-a)}(\lambda) = (1 - \mathrm{e}^{-\mathrm{i}\lambda})^{-a}$.

The output of the so defined fractional integration operator is a generalized process with generalized spectral measure ν' defined on V_{d+a} by

$$\mathrm{d}\nu'(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2a}\mathrm{d}\nu(\lambda).$$

The process Y can thus be equivalently defined by its spectral representation

$$Y^*(g) = X^*(\Delta^{(-a)} \times g), \quad g \in L^2(\nu') \ .$$

When $a + d < 1/2$ then ν' is a finite measure and Y is associated to a weakly stationary process $Y_t = Y^*(\psi_t)$, with $\psi_t(\lambda) = \mathrm{e}^{-\mathrm{i}\lambda t}$. When $a + d \geq 1/2$, the question arises about the existence of a discrete time process $\{Y_t,\, t \in \mathbb{Z}\}$ corresponding to the generalized process $\{Y(\mathrm{h}),\, \mathrm{h} \in V_{d+a}\}$, i.e. such that (7) holds with Y replacing X. The answer is given by Lemma 2.5: let k be the positive integer such that $k - 1/2 \leq d < k + 1/2$; then it is necessary and sufficient to have

$$[\boldsymbol{\Delta}^k Y]_t = Y^*(\Delta^{(-k)}\psi_t), \quad \text{where} \quad \psi_t(\lambda) = \mathrm{e}^{-\mathrm{i}\lambda t} \ .$$

There is a unique solution Y to this equation that satisfies $Y_0 = 0$ a.s. The whole set of solutions is then obtained by adding an arbitrary random polynomial of degree $k - 1$.

Definition 3.6 allows to define a fractional integration operator which preserves the property of increment stationarity, in contrast with the truncated fractional integrated process. Indeed consider an $M(d)$ process X and an arbitrary exponent $a \in \mathbb{R}$. Then $\boldsymbol{\Delta}^a X$ is an $M(d+a)$ process. Starting with a short memory process, $d = 0$, we obtain a class of *fractionally integrated processes* of arbitrary order, that include ARIMA and FARIMA processes for which we have $a \in \mathbb{Z}_+$ and $a < 1/2$ respectively.

3.3 A non-linear case: the infinite source Poisson process

We consider the infinite source Poisson transmission process defined by

$$(13) \quad X_t = \sum_{\ell \in \mathbb{Z}} W_\ell \, \mathbb{1}_{\{\Gamma_\ell \leq t < \Gamma_\ell + Y_\ell\}}, \quad t \in \mathbb{R} \ ,$$

where the triples $\{(\Gamma_\ell, Y_\ell, W_\ell),\ \ell \in \mathbb{Z}\}$ of session arrival times, durations and transmission rates satisfy

Assumption 1. (i) The arrival times $\{\Gamma_\ell,\ \ell \in \mathbb{Z}\}$ are the points of a homogeneous Poisson process on the real line with intensity λ, indexed in such a way that $\cdots < \Gamma_{-2} < \Gamma_{-1} < \Gamma_0 < 0 < \Gamma_1 < \Gamma_2 < \cdots$

(ii) The durations and transmission rates $\{(Y, W), (Y_\ell, W_\ell),\ \ell \in \mathbb{Z}\}$ are independent and identically distributed random pairs with values in $(0, \infty) \times [0, \infty)$ and independent of the arrival times $\{\Gamma_\ell,\ \ell \in \mathbb{Z}\}$.

(iii) There exist a real number $\alpha \in (0, 2)$ and a positive function L_2 slowly varying at infinity such that for all $t > 0$, $H_2(t) = \mathbb{E}\left[W^2 \mathbb{1}_{\{Y > t\}}\right] = L_2(t)t^{-\alpha}$.

These assumptions are used in Faÿ et al. [2007] to define a non-linear long range dependent process with long memory parameter

$$d = 1 - \alpha/2 \ .$$

In fact the process in (13) is only defined for $\mathbb{E}[Y] < \infty$, which implies $\alpha \geq 1$, see [Faÿ et al., 2007, Proposition 2.1].

If $\mathbb{E}[Y] = \infty$, there are again two ways to adapt the definition of X that correspond to the two ways of defining the fractional integration in a non-stationary context :

1. either truncate the sum in (13), which amounts to define a non-stationary process

$$X_t^{NS} = \sum_{\ell \geq 1} W_\ell \, \mathbb{1}_{\{\Gamma_\ell \leq t < \Gamma_\ell + Y_\ell\}}, \quad t \in \mathbb{R} \, ,$$

2. or define a generalized process $X(\theta)$, by

$$X(\theta) = \sum_{\ell \in \mathbb{Z}} W_\ell \left(\sum_{t \in I} \theta_t \mathbb{1}_{\{\Gamma_\ell \leq t < \Gamma_\ell + Y_\ell\}} \right) \, ,$$

for any sequence $\theta = (\theta_t)_{t \in I}$ with I a finite subset of $\mathbb{R}$, such that its Fourier transform

$$\theta^*(\xi) = \sum_{t \in I} \theta_t \, \mathrm{e}^{-\mathrm{i}t\xi} \quad \text{satisfies} \quad \int_{-1}^{1} |\theta^*(\xi)|^2 |\xi|^{-2d} \mathrm{d}\xi < \infty \, ,$$

which holds for any θ if $d < 1/2$ (that is, $\alpha \in (1,2)$) and is equivalent to $\sum_{t \in I} \theta_t = 0$ if $d \in [1/2, 1)$ (that is, $\alpha \in (0,1]$).

In the second case, it can be shown that $X(\theta)$ is well defined and entirely determined by $\tilde{X}_t = X(\theta^{(t)})$, $t \in \mathbb{R}$, with $\theta^{(t)}$ denoting the sequence with support $\{0, t\}$ defined by $\theta_0^{(t)} = -1$ and $\theta_t^{(t)} = 1$. Moreover a continuous time process $\{X_t, \, t \in \mathbb{R}\}$ is such that $X(\theta) = \sum_{t \in I} \theta_t X_t$ for any such θ if and only if $X_t - X_0 = \tilde{X}_t$ for all $t \in \mathbb{R}$.

Following Lemma 3.1 in Faÿ et al. [2007], we easily obtain that for θ as above

$$\mathrm{var}(X(\theta)) = \int |\theta^*(\xi)|^2 f(\xi) \, \mathrm{d}\xi \, , \quad \text{where} \quad f(\xi) = \frac{1}{4\pi} \xi^{-2} \mathbb{E}[W^2 (1 - \cos(Y\xi)]$$

can be interpreted as the generalized spectral density of the generalized process X.

Again we are in a case where the approach using generalized processes offers a more natural extension to the stationary situation than the approach using truncation.

4 Conclusion

As claimed in Section 3, generalized processes seems more appropriate to deal with non-stationary long memory processes than truncating the operator generating long memory (e.g. the fractional integration operator). The basic reason is that this approach includes the stationary case allowed for $d < 1/2$. On the other hand the idea that long range dependence can be viewed as some transition between short memory $d = 0$ and non-stationarity, $d \geq 1/2$ seems less appealing in this setting. In the setting based on generalized process for defining processes with long memory parameter $d \in \mathbb{R}$, it seems more appropriate to distinguish *entire* long memory parameter $d \in \mathbb{Z}$ from *fractional* long memory parameters $d \in \mathbb{R} \setminus \mathbb{Z}$.

Nevertheless the question of non-stationarity in the operator generating long memory remains of interest. We may look in this direction further away than a simple truncation that amounts to assume that the observed signal more or less starts in a null state at the origin. Instead assume that the operator changes along time, resulting in a change of the memory parameter as well. This idea were recently investigated in Roueff and von Sachs [2011]. A locally stationary model similar to that in Dahlhaus [1996] is defined. Here, however, the *local spectral density* at rescaled time $u \in [0,1]$ is of the form $f(u, \lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2d(u)} \, f^*(u, \lambda)$ with $\lambda \mapsto f^*(u, \lambda)$ denoting a short-memory spectral density, see Roueff and von Sachs [2011] for a detailed description of this model. Wavelet estimators can be adapted in this context to construct estimators of the local long memory parameter. The main difficulty here is to cope with two opposite goals: 1) provide a local analysis 2) focus on low frequencies.

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