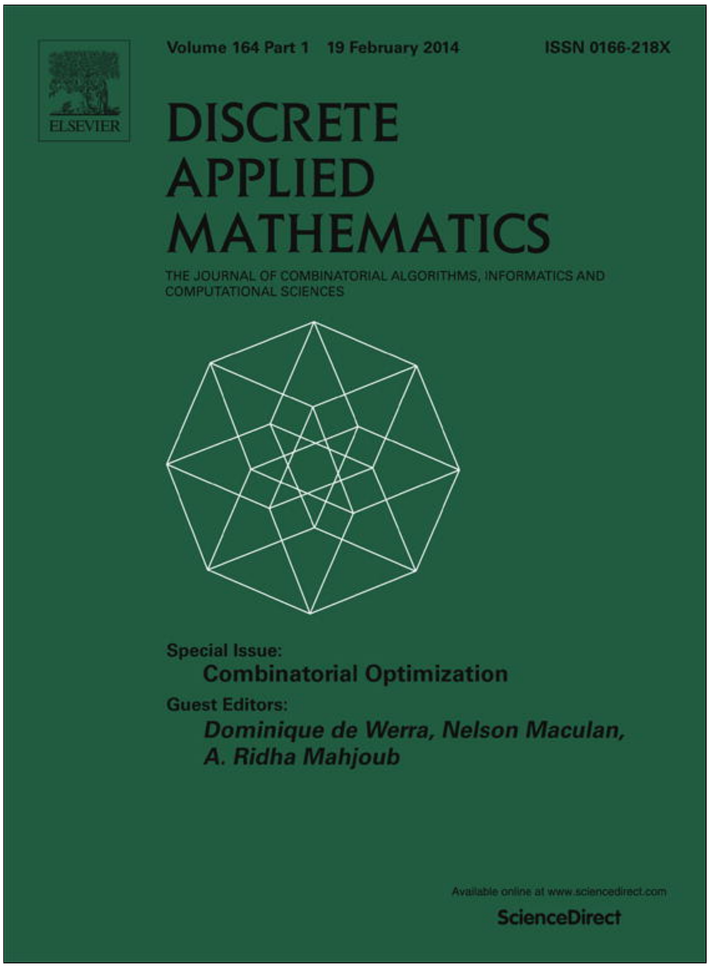
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Maximum size of a minimum watching system and the graphs achieving the bound

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ABSTRACT

Let $G = (V(G), E(G))$ be an undirected graph. A watcher w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$. If a vertex v belongs to $A(w)$, we say that v is covered by w. Two vertices v_1 and v_2 in G are said to be separated by a set of watchers if the list of the watchers covering v_1 is different from that of v_2. We say that a set W of watchers is a watching system for G if every vertex v is covered by at least one $w \in W$, and every two vertices v_1, v_2 are separated by W. The minimum number of watchers necessary to watch G is denoted by $w(G)$. We give an upper bound on $w(G)$ for connected graphs of order n and characterize the trees attaining this bound, before studying the more complicated characterization of the connected graphs attaining this bound.

1. Introduction

Let $G = (V(G), E(G))$ be an undirected connected graph (the case of an unconnected graph can also be treated, by considering its connected components separately). A *watcher* w of G is a couple $w = (\ell(w), A(w))$, where $\ell(w)$ belongs to $V(G)$ and $A(w)$ is a set of vertices of G at distance 0 or 1 from $\ell(w)$; in other words, $A(w)$ is a subset of $B(\ell(w))$, the ball of radius 1 centred at $\ell(w)$. We will say that w is *located* at $\ell(w)$ and that $A(w)$ is its *watching area* or *watching zone*. If a vertex v belongs to $A(w)$, we say that v is *covered* by w.

Two vertices v_1 and v_2 in G are said to be *separated* by a set of watchers if the list of the watchers covering v_1 is different from that of v_2.

We say that G is *watched* by a set W of watchers, or that W is a *watching system* for G, if,

- for every v in $V(G)$, there exists $w \in W$ such that v is covered by w;
- if v_1 and v_2 are two vertices of G, v_1 and v_2 are separated by W.

Note that several watchers can be located at the same vertex, and a watcher does not necessarily cover the vertex where it is located.

The minimum number of watchers necessary to watch a graph G is denoted by $w(G)$.

We will often represent watchers simply by integers, as for the graph G_0, which has 8 vertices, represented in Fig. 1: the location of a watcher is written inside a rectangle; for each vertex v of the graph, the list of watchers covering v is written in italics, so that the watching area of each watcher can be retrieved. In the example of Fig. 1, watcher 1 is located at c and covers the vertices a, c and d, watcher 2 is also located at c and covers the vertices b, c and e, watcher 3 is located at f and

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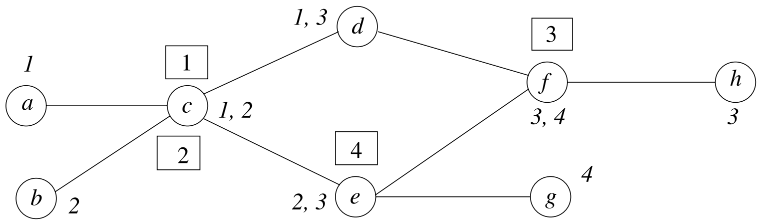

Fig. 1. A graph G_0 watched by watchers 1–4.

covers the vertices d, e, f and h, and watcher 4 is located at e and covers the vertices f and g. The graph G_0 is watched by these four watchers and, using inequality (1) below, we have that $w(G_0) = 4$.

Let G be a graph of order n. If we have a set W of k watchers, the number of distinct non-empty subsets of W is equal to $2^k - 1$. Therefore, it is necessary to have $2^k - 1 \geq n$, and so we have the following inequality:

$$w(G) \geq \lceil \log_2(n+1) \rceil. \tag{1}$$

Obviously, watching systems generalize *identifying codes* (see the seminal paper [10] and see [9] for a large bibliography): indeed, identifying codes are such that, for any $w = (\ell(w), A(w)) \in W$, we have

$$A(w) = B(\ell(w)),$$

which means that, in this case, a watcher, or *codeword*, necessarily covers itself and all its neighbours.

See also [8,11] for similar ideas.

Watching systems were introduced in [1,2], where motivations are exposed at large, basic properties are given, a complexity result is established, and the case of the paths and cycles is studied in detail, with comparison to identifying codes.

In Section 2, we give an upper bound on $w(G)$ when G is a connected graph with n vertices. In Section 3, we characterize the trees of order n which attain this bound: Theorems 7, 12 and 13 are for the cases $n = 3k$, $n = 3k+2$ and $n = 3k+1$, respectively. This helps us to study, in Section 4, the characterization of *maximal* graphs reaching the bound, that is, graphs to which no edge can be added without decreasing the minimum number of necessary watchers: Theorems 15 and 16 give the answer for $n = 3k$ and $n = 3k+2$ respectively, and Proposition 17 and Conjecture 18 are stated for the case $n = 3k+1$. This in turn gives results for all the connected graphs attaining the bound.

2. The maximum of minimum number of watchers

The following three easy lemmata will prove efficient. We recall that $H = (V(H), E(H))$ is a partial graph of $G = (V(G), E(G))$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

Lemma 1. *Let G be a graph, and let H be a partial graph of G. Then*

$$w(H) \geq w(G).$$

Proof. If H is watched by a set W of watchers, the same set W watches G, since two adjacent vertices in H are also adjacent in G. □

Note that this monotonicity property does not hold in general for identifying codes.

Lemma 2. *Let T be a tree, let x be a leaf of T, and let y be the neighbour of x.*

(a) *There exists a minimum watching system for T with one watcher located at y.*
(b) *If y has degree 2, there exists a minimum watching system for T with one watcher located at z, the second neighbour of y.*

Proof. (a) A watching system must cover x, so there is a watcher w_1 located at x or y, with $x \in A(w_1)$. If $w_1 = (x, A(w_1))$, then we can replace it by $w_2 = (y, A(w_1))$, since $B_1(y) \supseteq B_1(x)$.

(b) If, in a watching system of T, there is no watcher(s) located at z, then there are at least two watchers whose locations are in the set $\{x, y\}$. In the best case, these watchers cover x, y and z, and separate them pairwise. This task can just as well be done by two watchers located at y and z. □

Lemma 3. *Let T be a tree of order 4, and let v be a vertex of T; there exists a set W of two watchers such that*

- *the vertices in $V(T) \setminus \{v\}$ are covered and pairwise separated by W—in this case, we shall say, with a slight abuse of notation, that $V(T) \setminus \{v\}$ is watched by W;*
- *the vertex v is covered by at least one watcher.*

Proof. In Fig. 2, we give all possibilities: the two trees of order 4, and for each of them, the two locations for v (v is a leaf, or v is not a leaf). □

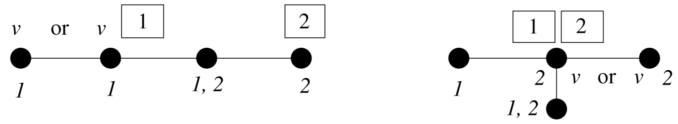

Fig. 2. Trees of order 4.

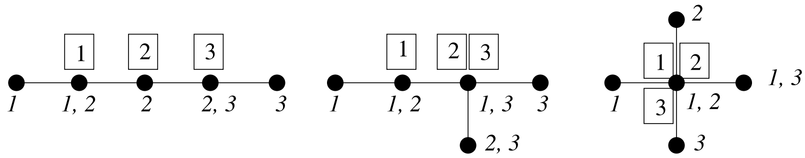

Fig. 3. The case $n = 5$ in Theorem 4.

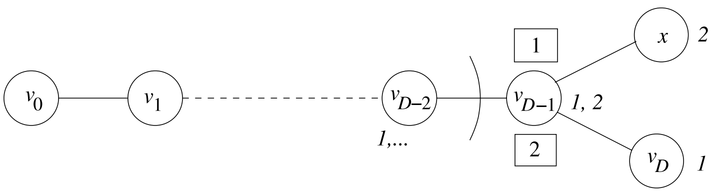

Fig. 4. First case of Theorem 4: the degree of v_{D-1} is equal to 3.

We are now ready to give an upper bound for $w(G)$ with respect to n, the order of G. Note in contrast that the upper bound for identifying codes, when they exist, is $n - 1$, see [4,7], and is reached, among other graphs, by the star; see also [5,6].

Theorem 4. *Let G be a connected graph of order n.*

- *If $n = 1$, $w(G) = 1$.*
- *If $n = 2$ or $n = 3$, $w(G) = 2$.*
- *If $n = 4$ or $n = 5$, $w(G) = 3$.*
- *If $n \notin \{1, 2, 4\}$, $w(G) \le \frac{2n}{3}$.*

The proof can be found in [1,2], but we give it here, because the results of the four cases into which it is divided will be frequently used in what follows.

Proof. For $n = 1$, $n = 2$, or $n = 3$, the result is direct. For $n = 4$, it is necessary to have at least $\lceil \log_2(5) \rceil = 3$ watchers, and it is easy to verify that this is sufficient. For $n = 5$, all possibilities are given by Fig. 3, and we can see that we always have $w(G) = 3$.

We proceed by induction on n. We assume that $n \ge 6$ and that the theorem is true for any connected graph of order less than n.

Let G be a connected graph of order n. Let T be a spanning tree of G; we will prove that $w(T) \le \frac{2n}{3}$ and then the theorem will result from Lemma 1. We denote by D the diameter of T (i.e., the maximum distance between two distinct vertices of T), and we consider a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ of T, with D edges.

We distinguish between four cases, according to some conditions on the degrees of v_{D-1} and v_{D-2}.

• *First case: the degree of v_{D-1} is equal to 3*

The vertex v_{D-1} is adjacent to a vertex x other than v_{D-2} and v_D; because D is the diameter, clearly x and v_D are leaves of T (see Fig. 4). We consider the tree obtained by removing x, v_{D-1} and v_D from T; this new tree T' has order $n - 3$.

If $n \ge 8$ or if $n = 6$, we consider a minimum set W of watchers watching T'; if $n = 7$, then T' is of order 4, and, using Lemma 3, we choose a set W of two watchers to watch $V(T') \setminus \{v_{D-2}\}$ *and* cover the vertex v_{D-2}.

Then, for T, in both cases, we add to W two watchers $w_1 = (v_{D-1}, \{v_{D-2}, v_{D-1}, v_D\})$ and $w_2 = (v_{D-1}, \{v_{D-1}, x\})$. In Fig. 4, we rename these watchers 1 and 2. Then T is watched by $W \cup \{w_1, w_2\}$. So, $w(T) \le |W| + 2 \le w(T') + 2$.

Now we use the induction hypothesis: if $n \ge 8$ or $n = 6$, then $w(T) \le \frac{2}{3}(n - 3) + 2 = \frac{2n}{3}$; and if $n = 7$, then $w(T) \le 2 + 2 = 4 < \frac{2}{3} \times 7$.

• *Second case: the degrees of v_{D-1} and v_{D-2} are equal to 2*

The neighbours of v_{D-1} are v_{D-2} and v_D, and the neighbours of v_{D-2} are v_{D-3} and v_{D-1} (see Fig. 5). We consider the tree obtained by removing v_{D-2}, v_{D-1} and v_D from T; this new tree T' has order $n - 3$.

If $n \ge 8$, or if $n = 6$, we consider a minimum set W of watchers watching T'; if $n = 7$, T' is of order 4; again using Lemma 3, we choose a set W of two watchers to watch $V(T') \setminus \{v_{D-3}\}$ and cover the vertex v_{D-3}. As in the first case, we add to W two watchers: $w_1 = (v_{D-2}, \{v_{D-3}, v_{D-2}, v_{D-1}\})$ and $w_2 = (v_{D-1}, \{v_{D-2}, v_D\})$, and T is watched. So, $w(T) \le |W| + 2 \le w(T') + 2$. The end of this case is the same as in the first case.

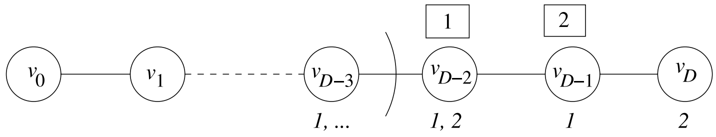

Fig. 5. Second case of Theorem 4: the degrees of v_{D-1} and v_{D-2} are equal to 2.

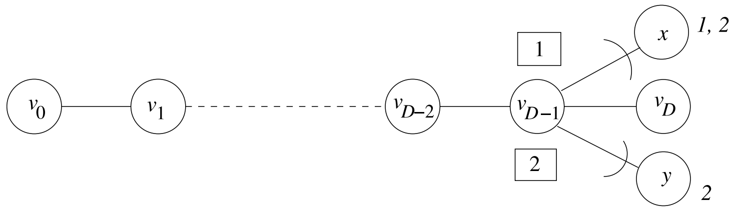

Fig. 6. Third case of Theorem 4: the degree of v_{D-1} is at least 4.

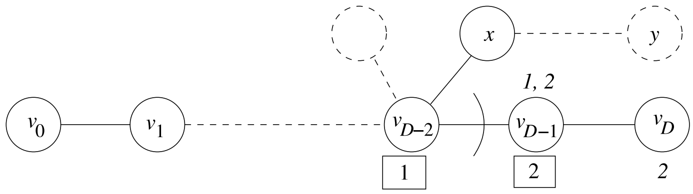

Fig. 7. Fourth case of Theorem 4: the degree of v_{D-1} is equal to 2 and the degree of v_{D-2} is at least 3.

• *Third case: the degree of* v_{D-1} *is at least 4*

The vertex v_{D-1} is adjacent to at least two vertices other than v_{D-2} and v_D: let x and y be two neighbours of v_{D-1} distinct from v_{D-2} and v_D; these two vertices are leaves of T (see Fig. 6). We consider the tree T' obtained by removing x and y from T. By Lemma 2, there exists a minimum set W of watchers watching T' with a watcher w_1 located at v_{D-1}. For T, we take the set W and we add the watcher $w_2 = (v_{D-1}, \{x, y\})$; we also add the vertex x to the watching area of w_1. Since the tree T' is watched by W, the tree T is watched by $W \cup \{w_2\}$. So, $w(T) \le w(T') + 1$.

If $n \ge 7$, the order of T' is at least 5 and, using the induction hypothesis, $w(T) \le \frac{2}{3}(n-2) + 1 < \frac{2n}{3}$.

If $n = 6$, then $n - 2 = 4$ and $w(T) \le 3 + 1 = 4 = \frac{2}{3} \times 6$.

• *Fourth case: the degree of* v_{D-1} *is equal to 2 and the degree of* v_{D-2} *is at least 3*

The neighbours of v_{D-1} are v_{D-2} and v_D. The vertex v_{D-2} is adjacent to v_{D-3} and v_{D-1} but also to at least one other vertex x (see Fig. 7); if the degree of x is at least 3, using the fact that the diameter of T is equal to D, we can use the first or third case to conclude the claim, with x playing the part of v_{D-1}.

So, we assume that the degree of x is 1 or 2; if its degree is 2, it has a neighbour y other than v_{D-2}.

We consider the tree T' of order $n - 2$ obtained by removing v_{D-1} and v_D from T. By Lemma 2, there exists a minimum set W of watchers watching T' with a watcher w_1 located at v_{D-2}. To watch T, we take the set W and add the watcher $w_2 = (v_{D-1}, \{v_{D-1}, v_D\})$; we also add the vertex v_{D-1} to the watching area of w_1. Then T is watched by $W \cup \{w_2\}$.

The end of this case is exactly the same as in the previous case. □

Remark 5. In the proof of Theorem 4, we have constructed, according to the cases, a tree T' with order $n - 3$ such that $w(T) \le w(T') + 2$, or a tree T' with order $n - 2$ such that $w(T) \le w(T') + 1$.

These two constructions, from T to T', will be used several times in what follows, for example, in the proof of Theorem 7.

3. Trees T of order n for which $w(T) = \lfloor \frac{2n}{3} \rfloor$

In this section, we characterize the trees T with n vertices and $w(T) = \lfloor \frac{2n}{3} \rfloor$. Our study is divided into three cases: $n = 3k$, $n = 3k + 2$ and $n = 3k + 1$. Note that the number of isomorphic trees of a given order can be determined: see, e.g., [12, Fig. 4, seq. 299] or [13]. The sequence goes 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, . . .; see also Figs. 10, 13 and 23.

We first define some particular trees, of orders 1–5, that we name *gadgets*. For each gadget, we choose one or two particular vertices named *binding vertices*, through which the different gadgets will be exclusively connected between themselves; a vertex which is not a binding vertex is said to be *ordinary*. In what follows, we will sometimes denote a gadget of order i by gi, $1 \le i \le 5$, and use the abbreviation b.v. for binding vertex. The gadgets are depicted in Fig. 8; we represent the binding vertices with squares and ordinary vertices with circles.

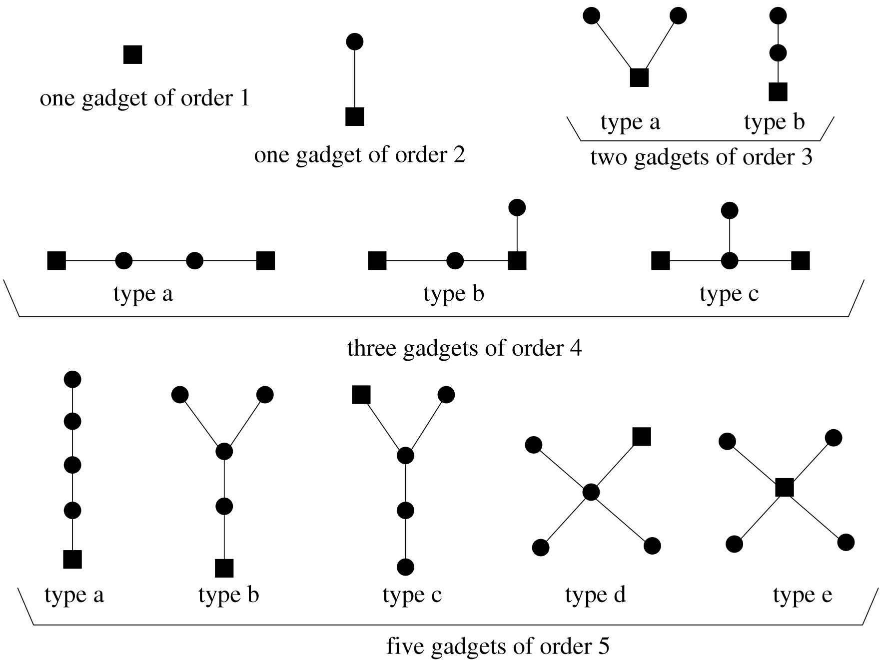

Fig. 8. All the gadgets.

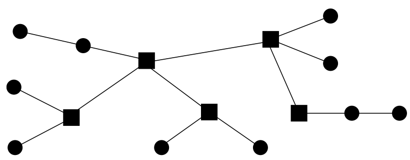

Fig. 9. The tree T_{15}.

We will use the following easy lemma, whose proof we omit.

Lemma 6. *Let T be a tree of order 3, and let v and v' be two distinct vertices in T. It is possible to watch T with one watcher located at v and one watcher located at v'.*

As a consequence, if T' is a tree of order 4 and x is a leaf of T', there exists a set W of two watchers such that $V(T') \setminus \{x\}$ is watched by W and x is covered by W.

The following theorem characterizes the trees T with order $n = 3k$ and $w(T) = 2k$.

Theorem 7. *Let T be a tree of order $n = 3k$ for $k \geq 1$. We have that $w(T) = 2k \Leftrightarrow T$ can be obtained by choosing k gadgets of order 3 and joining these gadgets by their binding vertices to obtain a tree.*

The tree T_{15} in Fig. 9 is an example of a tree reaching this maximum.

Proof. Assume that a tree T of order $n = 3k$ is obtained by choosing k gadgets of order 3 and joining these gadgets by their b.v.'s to form a tree. It is clear that, to watch T, it is necessary to locate two watchers on each gadget. So T reaches the bound $2k$.

We will prove the converse by induction on k. For $k = 1$, it is immediate. We also examine the case $k = 2$, that is to say, $n = 6$. We draw in Fig. 10 the six different trees T on six vertices; when a tree is not of a type described in the right part of the equivalence, we explicitly give the watchers showing that $w(T) = 3$ and, in the other cases, we simply indicate the b.v.'s of the two gadgets involved.

We will sometimes represent a $g3$ of type a or b with a triangle, as in Fig. 11: a dashed edge means that the edge may exist or not, with always exactly two edges in each $g3$. A watcher indicated inside the triangle means that this watcher is located at one of the three vertices of the triangle, at an appropriate vertex according to the case.

We assume now that $k \geq 3$ and that the theorem is true for $k' < k$. Let T be a tree of order $n = 3k$ with $w(T) = 2k$.

We consider again the proof of Theorem 4 using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ with D edges, where D is the diameter of T. Here, the third and fourth cases are impossible, because they imply that $w(T) < \frac{2n}{3} = 2k$, unless $n = 6$, which has just been dealt with. In the first case of Theorem 4, we rename the vertices v_{D-1}, v_D, x and v_{D-2} by a, b, c and d, respectively; in the second case, we rename the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3} by a, b, c and d, respectively; in both cases, we remove the vertices a, b and c from T and obtain a tree T' of order $3(k-1)$; by Remark 5, it appears that T' needs at least $w(T)-2 = 2k-2$ watchers and so $w(T') = 2(k-1)$ and we can apply the induction hypothesis to T': the vertex d belongs to a $g3$, say g.

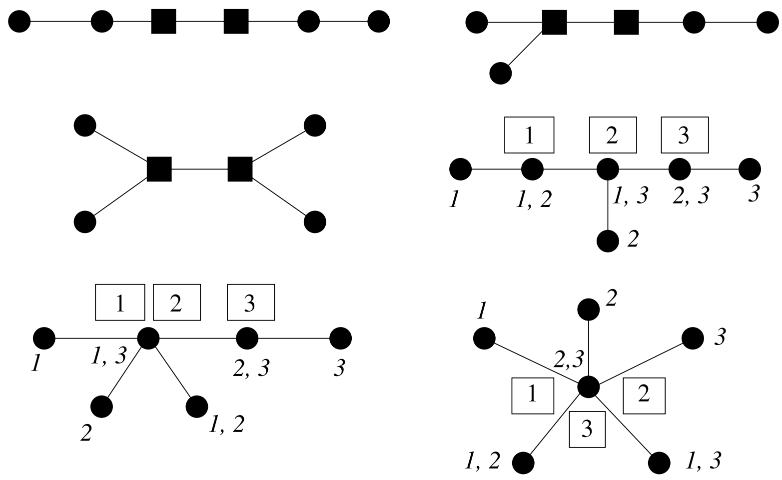

Fig. 10. The trees of order 6 for the proof of Theorem 7.

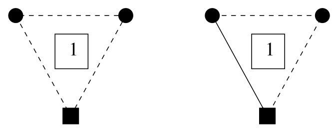

Fig. 11. Two representations for a $g3$ of type a or b.

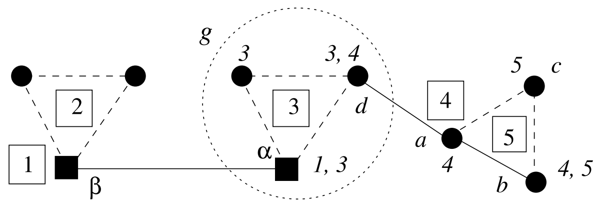

Fig. 12. $2k-1$ watchers are sufficient in T (end of proof of Theorem 7).

Assume that d is not the binding vertex of g. The b.v. α of g is adjacent to the b.v. β of another $g3$ in T' (see Fig. 12). By Lemma 6, we can locate watchers w_4 and w_1 at a and β, so that d is covered by w_4 and α is covered by w_1; it is then possible to watch T with only one watcher located on the gadget g, as we can see in Fig. 12, by choosing the appropriate vertex of g at which we locate the watcher denoted by 3. This leads to a contradiction on $w(T)$, and shows that d is the b.v. of g, in which case the result is immediately obtained, since $\{a, b, c\}$ can be seen as a $g3$, with its b.v. in a, connected to d. □

The following lemmata and definition will be used repeatedly in what follows.

Lemma 8. *Let T be a tree of order 5, and let v be a vertex of T. It is possible to watch T with three watchers, one of the three watchers being located at v.*

As a consequence, if T' is a tree of order 6 and x is a leaf of T', there exists a set W of three watchers such that $V(T') \setminus \{x\}$ is watched by W and x is covered by W.

Proof. The result for T is straightforward, by examining all the different possibilities, as we can see in Fig. 13; the consequence on T' is immediate. □

Lemma 9. *Consider a $g5$ with binding vertex α and ordinary vertices v, x, y and z; there exists a set W of two watchers such that*

- *$\{x, y, z\}$ is watched by W;*
- *the vertex v is covered by W.*

Proof. If the $g5$ is of type a, b, c, or d, then the four vertices v, x, y, z form a tree, and, by Lemma 3, we are done. If the $g5$ is of type e, then it is also possible, with two watchers located at α, the centre of the star, to watch $\{x, y, z\}$ and cover v. □

Definition 10. Let $H = (V(H), E(H))$ be a connected graph, and let v be a vertex in $V(H)$; let H' be the graph obtained by removing the vertex v from H (H' is connected or not). We say that v is free of charge, or free, in H if there exists a minimum watching system for the graph H' which is also a watching system for H.

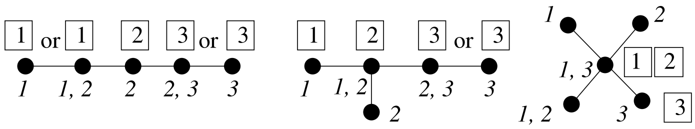

Fig. 13. Illustration for Lemma 8.

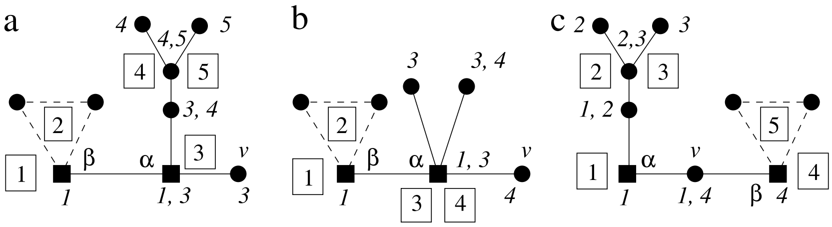

Fig. 14. Illustration for Lemma 11.

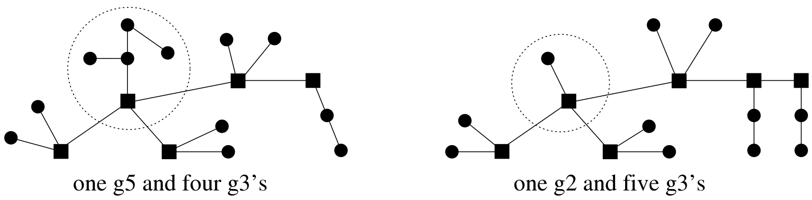

Fig. 15. The trees T_{17} and T'_{17}.

Lemma 11. *Let p be an integer satisfying $p \geq 2$. Let F be a forest obtained by choosing p gadgets of order 3 or 5 and possibly, if desired, by adding edges between the binding vertices of the p gadgets. Let v be a new vertex, which is adjacent to at least one binding vertex and cannot be adjacent to ordinary vertices; we assume that the graph obtained by adding v to F is a tree, say T. Then, the vertex v is free in T.*

Proof. If v is adjacent to only one b.v., let α be this vertex; since T is connected and $p \geq 2$, the vertex α is adjacent to another b.v., say β. If v is adjacent to at least two b.v.'s among the p gadgets, let α and β be two such vertices. Fig. 14 illustrates the lemma in detail in three cases:

(a) v is linked to the b.v. α of the $g5$ of type b and α is linked to the b.v. of a $g3$;
(b) v is linked to the b.v. α of the $g3$ of type a and α is linked to the b.v. of a $g3$;
(c) v is linked to the b.v.'s of the $g5$ of type b and of a $g3$.

After checking all types for the $g5$ in (a) and (c) and for the $g3$ in (b), one has checked the cases when (a) v is linked to a $g5$ linked to a $g3$; (b) v is linked to a $g3$ linked to a $g3$; (c) v is linked to a $g5$ and a $g3$. Using Lemmata 6 and 8 repeatedly, the remaining cases can be treated in exactly the same way. □

We are now ready to characterize the trees T with order $n = 3k + 2$ and $w(T) = 2k + 1$.

Theorem 12. *Let T be a tree of order $n = 3k + 2$ for $k \geq 1$. We have that $w(T) = 2k + 1 \Leftrightarrow T$ can be obtained by choosing one gadget of order 2 and k gadgets of order 3, or one gadget of order 5 and $k - 1$ gadgets of order 3, and joining these gadgets by their binding vertices to obtain a tree.*

The trees T_{17} and T'_{17} of Fig. 15 are examples of trees which attain this maximum.

Proof. Assume that a tree T of order $n = 3k + 2$ is obtained by choosing one $g2$ and k $g3$'s, or one $g5$ and $k - 1$ $g3$'s, and finally joining these gadgets by their binding vertices, in order to obtain a tree. It is necessary to locate one watcher on a $g2$, two watchers on a $g3$ and, because a $g5$ has four ordinary vertices, three watchers on a $g5$. So T attains the bound $2k + 1$: if there is a $g2$, we need one watcher for the $g2$ and $2k$ watchers for the k $g3$'s, if there is a $g5$, we need three watchers for the $g5$ and $2k - 2$ watchers for the $k - 1$ $g3$'s.

We will prove the converse by induction on k. For $k = 1$, $n = 5$ and the result is clear; see Fig. 3: T is a $g5$ (and in two out of three cases, it can also be seen as the connection of a $g2$ and a $g3$). We assume now that $k \geq 2$ and that the theorem is true for $k' < k$. Let T be a tree of order $n = 3k + 2$ with $w(T) = 2k + 1$. We consider again the proof of Theorem 4, using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ with D edges, where D is the diameter of T.

• *Part* (a): *we assume that we are in the first or second case in the proof of* Theorem 4

In the first case, we rename the vertices v_{D-1}, v_D, x and v_{D-2} by a, b, c and d, respectively; in the second case, we rename the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3} by a, b, c and d, respectively; we remove the vertices a, b and c from T and obtain a tree T' of

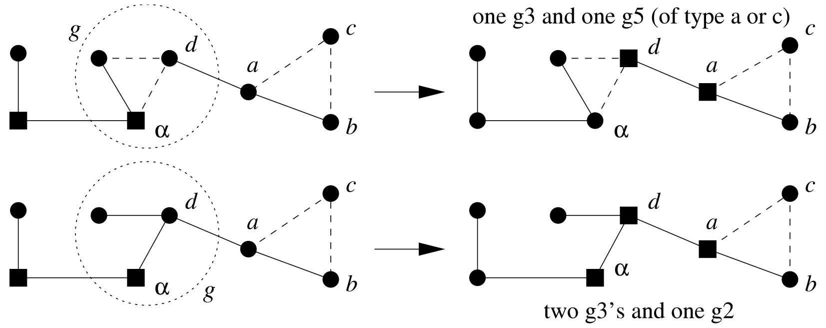

Fig. 16. Cases for $n = 8$ in part (a) of Theorem 12, when g is of order 3.

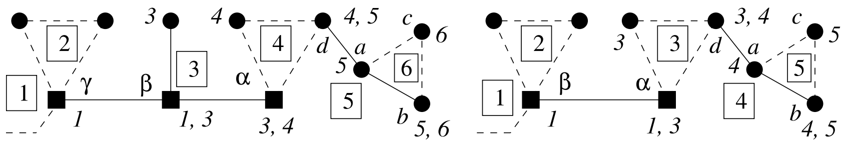

Fig. 17. Cases for $n \geq 11$ in part (a) of Theorem 12, when g is of order 3.

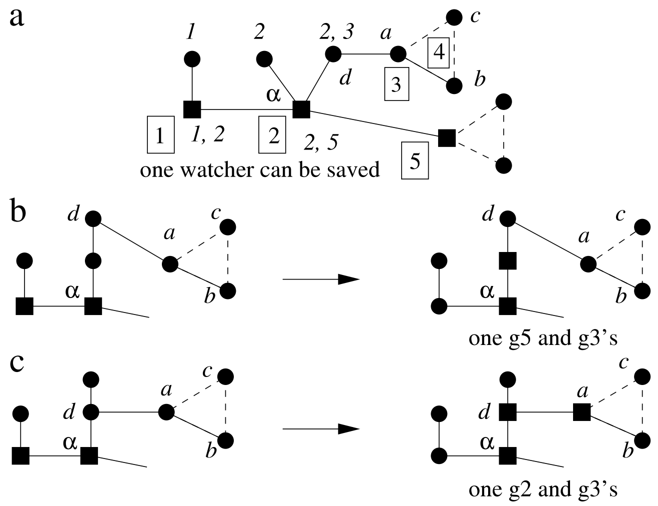

Fig. 18. More cases for $n \geq 11$ in part (a) of Theorem 12, when g is of order 3.

order $3(k-1)+2$; by Remark 5, it appears that T' needs at least $w(T) - 2 = 2k - 1$ watchers and so $w(T') = 2(k-1)+1$ and we can apply the induction hypothesis to T': T' is of one of the two types described in the right part of the equivalence, and the vertex d belongs to a gadget g, whose b.v. we denote by α. Assume first that $d \neq \alpha$.

- (i) If g is of order 2, then the subtree induced by the vertices of g and the vertices a, b and c yields a $g5$ of type a or b, and the result is proved for T.
- (ii) Assume next that g is of order 3. If T is of order 8, the two possibilities are given by Fig. 16. Assume therefore that T is of order at least 11.

Then there are four cases: (ii1) α is connected to a $g2$, which is itself connected to at least one more gadget, i.e., a $g3$; (ii2) α is not connected to a $g2$, but is connected to a $g3$; (ii3) α is not connected to a $g2$, but is connected to a $g5$; (ii4) α is connected to a $g2$ connected only to g.

In case (ii1), the left part of Fig. 17 shows how to use only one watcher for g, which leads to a contradiction on $w(T)$. In case (ii2), the same is true as shown by the right part of Fig. 17, which actually is the same as Fig. 12: notice that, since α is not connected to a $g2$, there does not have to be a watcher located at α. Case (ii3) goes through in exactly the same way as (ii2), using Lemma 8. The final case (ii4) is treated in Fig. 18.

• (iii) Finally, assume that g is of order 5. If T is of order 8, the reader will convince himself/herself that locating d at all the different vertices, except at the b.v. α, of all the different types for a $g5$ leads to the six patterns given by Fig. 19. If T is of order 11, the b.v. α is adjacent to the b.v. β of a $g3$. When one examines the different possibilities, it appears that, if T reaches the bound, it is of the desired shape: this is shown by Fig. 20.

To close the case when g is of order 5, we study the case when T is of order at least 14; then the tree T'' obtained from T' by removing the four vertices of g other than α has order at least 7 and we can apply Lemma 11 to it, which shows that the vertex α is free in T''. Using Lemma 9, we can use two watchers on g to watch $V(g) \setminus \{\alpha, d\}$ and cover the vertex d. With

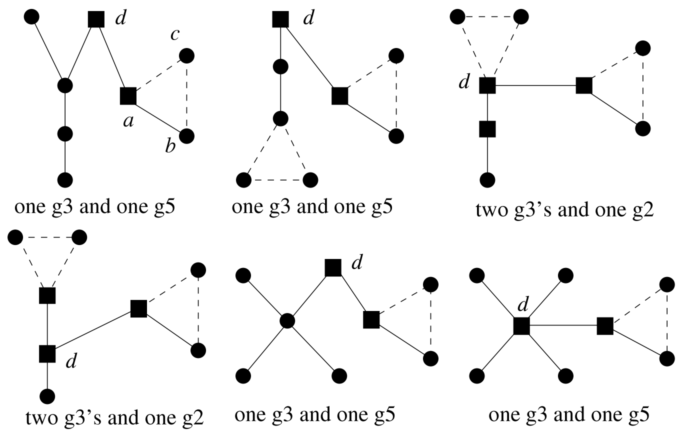

Fig. 19. Cases for $n = 8$ in part (a) of Theorem 12, when g is of order 5.

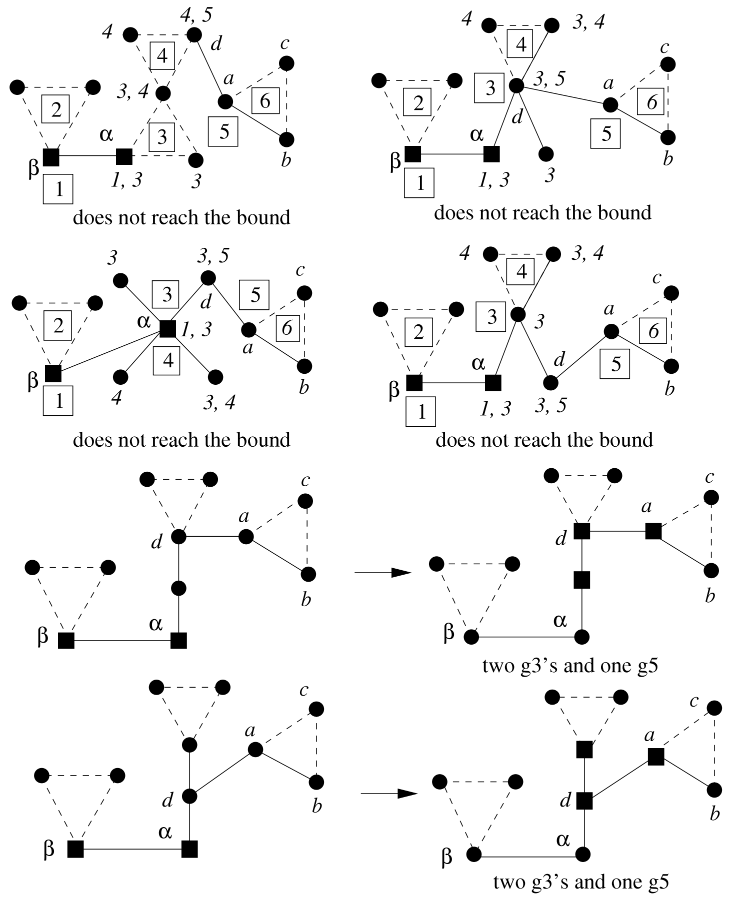

Fig. 20. Cases for $n = 11$ in part (a) of Theorem 12, when g is of order 5.

one watcher at a covering d, we can separate d from all the other vertices: so, we can do with only two watchers on g, and T does not attain the bound.

This shows that, if $d \neq \alpha$, then either the tree does not attain the bound, or it is of the desired form. On the other hand, if $d = \alpha$, then the result is immediately obtained. This ends part (a).

• *Part* (b): *we assume that we are in the third or fourth case in the proof of* Theorem 4

If we are in the third case, we remove the vertices x and y, and if we are in the fourth case, we remove the vertices v_{D-1} and v_D; we obtain a tree T' of order $3k$. By Remark 5, we have $w(T') = 2k$ and Theorem 7 may be used: T' can be obtained as a collection of $g3$'s linked by some edges between their binding vertices. So, the vertex v_0 is a leaf of a $g3$, say g; now we reverse the longest path $v_0, v_1, \ldots, v_D$ in T. If g is of type a (see the left part of Fig. 21), then v_{D-1} is linked to only one b.v., v_{D-2}, and has degree 3, because D is the diameter of the tree, and we are brought back to the first case. And if g is of type b, then v_{D-1} has degree 2, and either v_{D-2} has degree 2 and we are in the second case, or v_{D-2} has degree at least 3 and we are in the fourth case, with at least one b.v. x linked to v_{D-2} and x of degree at least 2 (see the right part of Fig. 21); however,

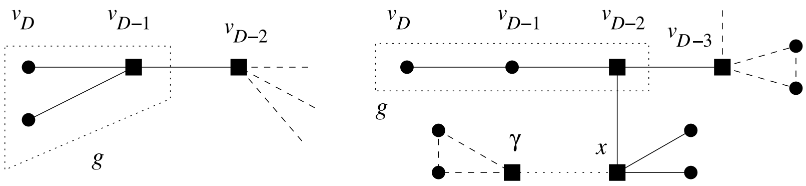

Fig. 21. Illustration for part (b) of Theorem 12.

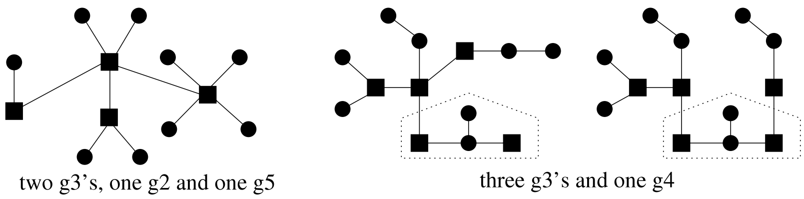

Fig. 22. The trees T_{13}, T'_{13} and T''_{13}.

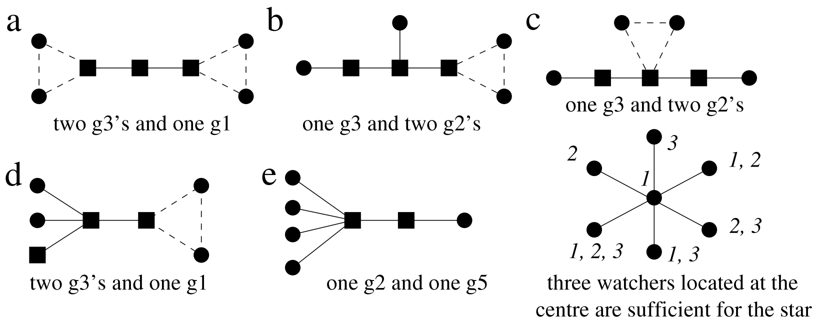

Fig. 23. All the possibilities for $n = 7$ in Theorem 13.

x cannot be linked to another b.v. γ, since this would increase the diameter of the tree, and for the same reason the $g3$ of x is of type a, so that necessarily x has degree 3. With x playing the part of v_{D-1}, we are again in the first case. In all cases, we can reuse the result obtained in part (a). □

The last case, $n = 3k + 1$ and $w(T) = 2k$, offers the greatest number of possibilities for the gadgets.

Theorem 13. *Let T be a tree of order $n = 3k + 1$ for $k \geq 2$. We have that $w(T) = 2k \Leftrightarrow T$ can be obtained by choosing*

- (i) *two gadgets of order* 2 *and* $k - 1$ *gadgets of order* 3*,*
- (ii) *or one gadget of order* 2*, one gadget of order* 5 *and* $k - 2$ *gadgets of order* 3*,*
- (iii) *or two gadgets of order* 5 *and* $k - 3$ *gadgets of order* 3*,*
- (iv) *or one gadget of order* 1 *and* k *gadgets of order* 3*,*
- (v) *or one gadget of order* 4 *and* $k - 1$ *gadgets of order* 3*,*

and joining these gadgets by their binding vertices to obtain a tree.

The trees T_{13}, T'_{13} and T''_{13} of Fig. 22 are examples of trees attaining the bound $2k$ for $n = 3k + 1$ (with $k = 4$).

Sketch of proof. Assume that a tree T of order $n = 3k+1$ is obtained as specified in the right part of the above equivalence. It is necessary to locate one watcher on a $g2$, two watchers on a $g3$ and two on a $g4$ (because a $g4$ has two ordinary vertices), and three watchers on a $g5$. So T reaches the bound $2k$; indeed, if

we are in (i), $(2 \times 1) + ((k - 1) \times 2) = 2k$;
in (ii), $(1 \times 1) + (1 \times 3) + ((k - 2) \times 2) = 2k$;
in (iii), $(2 \times 3) + ((k - 3) \times 2) = 2k$;
in (iv), $(1 \times 0) + (k \times 2) = 2k$;
in (v), $(1 \times 2) + ((k - 1) \times 2) = 2k$.

We will prove the converse by induction on k. For $n = 7$, the different possibilities are examined in Fig. 23. Now, we assume that $n \geq 10$.

We use the same scheme of proof as for Theorem 12: we assume that $k \geq 3$ and that the theorem is true for $k' < k$, we let T be a tree of order $n = 3k+1$ with $w(T) = 2k$, and we consider the proof of Theorem 4, using a path $v_0, v_1, v_2, \ldots, v_{D-1}, v_D$ with D edges, where D is the diameter of T.

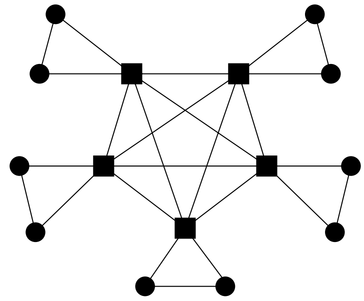

Fig. 24. G_{15}, the maximal graph of order 15 reaching the bound.

• *Part* (a): *we assume that we are in the first or second case in the proof of* Theorem 4

In the first case, we rename the vertices v_{D-1}, v_D, x and v_{D-2} by a, b, c and d, respectively; in the second case, we rename the vertices v_{D-2}, v_{D-1}, v_D and v_{D-3} by a, b, c and d, respectively. In both cases, we remove the vertices a, b and c from T and obtain a tree T' with order $3(k-1)+1$; by Remark 5, it appears that T' needs at least $w(T)-2=2k-2$ watchers and so $w(T')=2(k-1)$: we can apply the induction hypothesis to T', which is of one of the five types described in the right part of the equivalence.

The vertex d belongs to a gadget g; as before, if d is a binding vertex, we are done, so we assume from now on that d is ordinary, so that g is of order 2 or more, and we have four cases, according to the order of g.

• (1) If g is of order 2, the subtree induced by the vertices of g and the vertices a, b and c yields a $g5$ and the result is proved: indeed, if T' has two $g2$'s and $k-2$ $g3$'s (case (i)), or one $g2$, one $g5$ and $k-3$ $g3$'s (case (ii)), then T can be obtained with one $g2$, one $g5$ and $k-2$ $g3$'s (case (ii)), or two $g5$'s and $k-3$ $g3$'s (case (iii)), respectively.

- (2) g is of order 3.
- (3) g is of order 5.
- (4) g is of order 4.

The full treatment of these three cases can be found in [3].

• *Part* (b): *we are in the third or fourth case in the proof of* Theorem 4

The full treatment of this part can be found in [3]. □

4. Graphs G reaching the maximum value of $w(G)$

We first give the following definition.

Definition 14. A connected graph G is said to be maximal if, when we add any edge to G, we obtain a graph G' for which $w(G') < w(G)$.

We denote by $\omega(n)$ the maximum of minimum number of watchers needed in a connected graph of order n, i.e.,

$$\omega(n) = \max\{w(G) : G \text{ connected of order } n\}.$$

In the previous section, we have established that $\omega(n) = \lfloor \frac{2n}{3} \rfloor$ for $n \notin \{1, 2, 4\}$, and we have characterized the trees of order n reaching $\omega(n)$. In this section, we want to describe all the maximal connected graphs of order n which reach $\omega(n)$. Using Lemma 1, the graphs of order n which reach $\omega(n)$ are exactly the connected partial graphs of the maximal connected graphs of order n reaching $\omega(n)$.

We recall that K_p denotes the complete graph (or clique) of order p. Again, we divide our study into three cases, $n=3k$, $n=3k+2$ and $n=3k+1$.

Theorem 15. *Let k be an integer, $k \geq 1$, and let G be a maximal graph of order $3k$. We have that $w(G)=2k \Leftrightarrow G$ is obtained by taking a collection of k K_3's, choosing one vertex named a* binding vertex *in each K_3, and connecting these k binding vertices by K_k.*

For instance, the graph G_{15} of Fig. 24 is the unique maximal graph of order 15 reaching the bound $\omega(15)=10$.

Proof. The implication from the right to the left is direct. So, given a maximal graph G of order $3k$ satisfying $w(G)=2k$, we have to prove that G is of the form described in the theorem. Let T be a spanning tree of G. Using Lemma 1 and Theorem 4, we can see that $w(T)=2k$. By Theorem 7, T is a collection of k gadgets of order 3 connected by their binding vertices. We shall show that in G any edge which is not in T is an edge between two b.v.'s of T, or is the missing edge of a $g3$; to do this, we assume that there is in G an edge e which is not an edge between two b.v.'s of T, nor the missing edge of a $g3$. In Fig. 25, we consider the four possibilities:

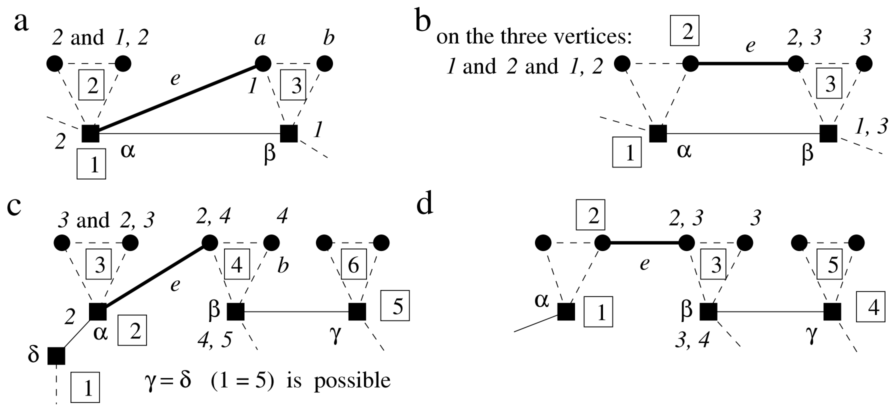

Fig. 25. Forbidden edges between two $g3$'s in the proof of Theorem 15.

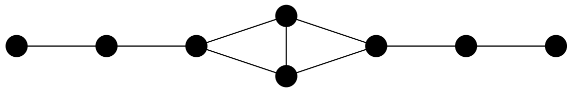

Fig. 26. A maximal graph of order 8 reaching the bound.

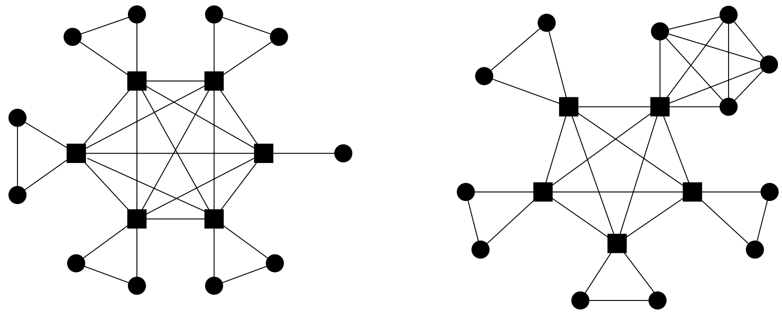

Fig. 27. The two maximal graphs of order 17 reaching the bound.

(a) The edge e links an ordinary vertex a of a $g3$, denoted by g_3, whose b.v. is denoted by β, and the b.v. α of another $g3$, and the edge $\{\alpha, \beta\}$ exists; then, whatever the type of g_3, we can locate a watcher 3 on g_3 covering a, b and α, and the six vertices are covered and separated by three watchers only.

(b) e links two ordinary vertices of two $g3$'s which are linked by their b.v.'s. Again, the six vertices involved can be watched by three watchers.

In passing, these two cases show how to handle the case $n = 6$, so from now on we assume that $n \geq 9$.

(c) e links an ordinary vertex of a $g3$, whose b.v. is β, and the b.v. α of another $g3$, and $\{\alpha, \beta\}$ does not exist. Then α and β are linked to at least one other $g3$ (possibly the same), because, in the spanning tree T, there is a connection between any two b.v.'s.

(d) This is also true when e links two ordinary vertices of two $g3$'s which are not linked by their b.v.'s.

In each of these two cases, we can see that we are able to locate only one watcher on a $g3$, so there is a contradiction with the value of $w(G)$.

Furthermore, if we add to T the missing edge on each $g3$ and all the missing edges between the b.v.'s of T, the number of needed watchers remains equal to $2k$: we have obtained the unique maximal graph containing T. □

Theorem 16. (a) *Let k be an integer, $k \geq 3$, and let G be a maximal graph of order $3k + 2$. We have that $w(G) = 2k + 1 \Leftrightarrow G$ is obtained by taking a collection of k K_3's and one K_2, or $k - 1$ K_3's and one K_5, choosing one vertex named a binding vertex in each of these complete graphs, and connecting these binding vertices by K_{k+1} if we have taken a K_2, and by K_k if we have taken a K_5.*

(b) *If G is a maximal graph of order 8, then we have that $w(G) = 5 \Leftrightarrow G$ is the graph given by Fig. 26, or G is obtained by following the rules given in Case (a), for $k = 2$.*

(c) *The only maximal graph G of order 5 with $w(G) = 3$ is the clique K_5.*

For instance, the graphs G_{17} and G'_{17} of Fig. 27 are the two maximal graphs of order 17 reaching the bound $\omega(17) = 11$.

Sketch of proof. The implications from the right to the left are direct. So, given a maximal graph G of order $3k+2$ satisfying $w(G) = 2k + 1$, we have to prove that G is of the form(s) described in the theorem.

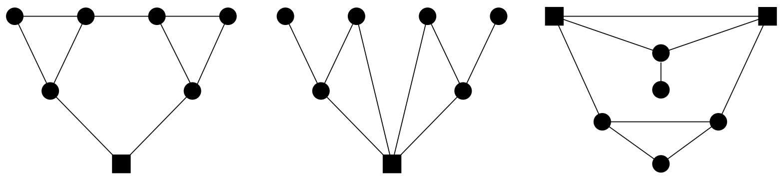

Fig. 28. The three new gadgets of order 7.

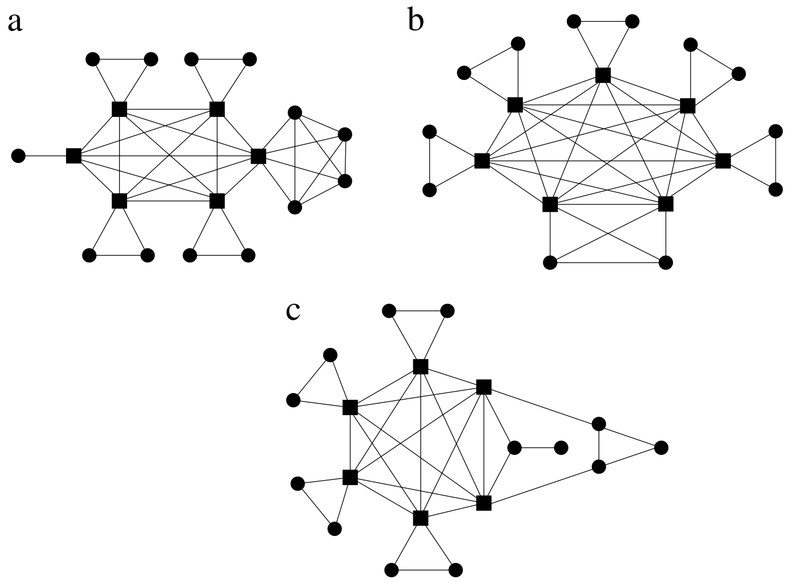

Fig. 29. Three graphs reaching the bound $\omega(19) = 12$.

By inequality (1) from the Introduction and Theorem 4, all connected graphs G of order 5 are such that $w(G) = 3$, K_5 is the unique maximal graph of order 5, and Case (c) is true.

The case $n = 8$, which does not fit the general framework either, is rather tedious to check, and is not given here.

We assume from now on that $n \geq 11$. Let T be a spanning tree of G. Using Lemma 1 and Theorem 4, we can see that $w(T) = 2k+1$. From Theorem 12, T can be obtained as one $g2$ or one $g5$ plus a collection of $g3$'s, with the gadgets connected by their binding vertices to form a tree. If, among the spanning trees of G, there is one with a $g5$, we choose this tree; and if, in all the spanning trees, we cannot avoid a $g2$, then we choose a tree in which the b.v. of the $g2$ has maximum degree (in the tree).

We shall list pairs of vertices which cannot be adjacent in the maximal graph G: between $g3$'s, between the $g5$ and a $g3$, and between the $g2$ and a $g3$ (the most delicate case).

All these cases are fully treated in [3]. Once this is done, we are in a position to conclude the following: if in T there are edges between ordinary vertices of different gadgets or between the b.v. of a gadget and an ordinary vertex of another gadget, then another spanning tree should have been chosen, containing a $g5$ instead of a $g2$, or containing a $g2$ with binding vertex of higher degree, or these edges are part of K_5, or we can save watchers.

Furthermore, if we add to T the missing edge on each $g3$, the missing edges on the possible $g5$, and all the missing edges between the b.v.'s in T, the number of needed watchers remains equal to $2k + 1$: we have obtained the only maximal graph containing T. □

Our complete proof of the previous theorem, for $n = 3k+2$, which is very long, including nine figures, is not very encouraging in view of the case $n = 3k + 1$. Indeed, although we have some insight into the situation, we can only give the following proposition and conjecture, in which, to describe the graphs, we need three new gadgets of order 7 (which are not trees), with one or two binding vertices; see Fig. 28. A $g7$ denotes a gadget of order 7. Unlike the $g4$'s, the rightmost $g7$ of the figure must have each of its two binding vertices connected to other b.v.'s in the following statement.

Proposition 17. *Let k be an integer, $k \geq 6$, and let G be a graph of order $3k + 1$ obtained by*

- (i) *taking two K_2's and $k - 1$ K_3's,*
- (ii) *or taking one K_2, one K_5 and $k - 2$ K_3's,*
- (iii) *or taking two K_5's and $k - 3$ K_3's,*
- (iv) *or taking one K_4 and $k - 1$ K_3's,*
- (v) *or taking one $g7$ and $k - 2$ K_3's,*

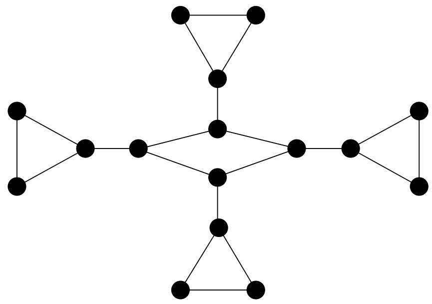

Fig. 30. A maximal graph reaching the bound $\omega(16) = 10$.

choosing one vertex named a binding vertex on each of the complete components K_i, except on K_4 for which we choose two binding vertices, taking for the g7 one or two binding vertices according to its type, and connecting these binding vertices to form a complete graph with them.

Then $w(G) = 2k$.

Proof. The proof is straightforward and is left to the reader. □

Conjecture 18. (1) *The graphs described in the previous proposition are maximal.*
(2) *They are the only maximal graphs attaining the bound $2k$.*

The graphs of Fig. 29 are examples of graphs described in Proposition 17. They have order 19 and reach the bound $\omega(19) = 12$: (a) with one K_2, one K_5 and four K_3's; (b) with one K_4 and five K_3's; (c) with one $g7$ and four K_3's.

For $n = 3k + 1$ with $k \leq 5$, there are maximal graphs needing $2k$ watchers which are not of the form described in Proposition 17. We give a certified example for $n = 16$ in Fig. 30.

References

[1] D. Auger, I. Charon, O. Hudry, A. Lobstein, Systèmes de contrôle dans les graphes: une extension des codes identifiants, Rapport Interne Télécom ParisTech-2010D011, Paris, France, p. 40, 2010 (in English).
[2] D. Auger, I. Charon, O. Hudry, A. Lobstein, Watching systems in graphs: an extension of identifying codes, Discrete Applied Math., in press (http://dx.doi.org/10.1016/j.dam.2011.04.025).
[3] D. Auger, I. Charon, O. Hudry, A. Lobstein, Maximum size of a minimum watching system and the graphs achieving the bound: full version, 2012. http://hal.archives-ouvertes.fr/hal-00680690.
[4] I. Charon, O. Hudry, A. Lobstein, Extremal cardinalities for identifying and locating-dominating codes in graphs, Rapport interne Télécom Paris-2003D006, Paris, France, p. 18, 2003.
[5] F. Foucaud, E. Guerrini, M. Kovše, R. Naserasr, A. Parreau, P. Valicov, Classifying graphs with minimum identifying code of size $n - 1$, Abstracts of the 8th French Combinatorial Conference, Orsay, France, No. 151, 2010.
[6] F. Foucaud, E. Guerrini, M. Kovše, R. Naserasr, A. Parreau, P. Valicov, Extremal graphs for the identifying code problem, Europ. J. Combin. 32 (2011) 628–638.
[7] S. Gravier, J. Moncel, On graphs having a $V \setminus \{x\}$ set as an identifying code, Discrete Math. 307 (2007) 432–434.
[8] I. Honkala, A. Lobstein, On identification in Z^2 using translates of given patterns, J. Universal Comput. Sci. 9 (10) (2003) 1204–1219.
[9] http://www.infres.enst.fr/~lobstein/bibLOCDOMetID.html.
[10] M.G. Karpovsky, K. Chakrabarty, L.B. Levitin, On a new class of codes for identifying vertices in graphs, IEEE Trans. Inform. Theory 44 (1998) 599–611.
[11] P. Rosendahl, On the identification of vertices using cycles, Electron. J. Combin. 10 (1) (2003) R7.
[12] N.J.A. Sloane, A Handbook of Integer Sequences, Academic Press, New York–San Francisco–London, 1973.
[13] N.J.A. Sloane, The on-line encyclopedia of integer sequences. https://oeis.org/A000055.