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Complexity results for extensions of median orders to different types of remoteness

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Abstract Given a finite set X and a collection $\Pi = (R_1, R_2, \ldots, R_v)$ of v binary relations defined on X and given a remoteness ρ, a relation R is said to be a central relation of Π with respect to ρ if it minimizes the remoteness $\rho(\Pi, R)$ from Π. The remoteness ρ is based on the symmetric difference distance $\delta(R_i, R)$ between R and the binary relations R_i of Π $(1 \leq i \leq v)$, which measures the number of disagreements between R_i and R. Usually, the considered remoteness between Π and a relation R is the remoteness $\rho_1(\Pi, R)$ given by the sum of the distances $\delta(R_i, R)$ over i, and thus measures the total number of disagreements between Π and R or, divided by v, provides the (arithmetical) mean number of disagreements between Π and R. The computation of a central relation with respect to ρ_1 is often an NP-hard problem when the central relation is required to fulfill structural properties like transitivity. In this paper, we investigate other types of remoteness ρ, for instance the sum of the pth power of the $\delta(R_i, R)$'s for any integer p, the maximum of the $\delta(R_i, R)$'s, the minimum of the $\delta(R_i, R)$'s, and different kinds of means of the $\delta(R_i, R)$'s, or their weighted versions. We show that for many definitions of the remoteness, including the previous ones, the computation of a central relation with respect to ρ remains an NP-hard problem, even when the number v of relations is given, for any value of v greater than or equal to 1.

Keywords Complexity · NP-completeness · Aggregation of preferences · Remoteness · Symmetric difference distance · Median relation · Median order · Binary relations · Complete preorder · Weak order · Linear order · Tournament · Slater's problem · Kemeny's problem

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1 Introduction

In an election, assume that we are given a finite set $X = \{1, 2, \dots, n\}$ of n candidates and a collection (or multi-set) $\Pi = (R_1, R_2, \dots, R_v)$, called a *profile*, of the preferences R_i of v voters ($1 \le i \le v$) who want to rank the n candidates. These preferences can be binary relations without noticeable properties, or can be linear orders, or complete preorders, and so on (see below for the definition of these ordered structures). Note that the relations involved in the profile may be the same: two different voters may share the same preference.

In order to aggregate these v relations into a collective ranking, one possibility consists in computing a relation belonging to a prescribed set $\mathcal{C}$ (for instance $\mathcal{C}$ is the set $\mathcal{L}$ of all the linear orders defined on X) which summarizes Π "as well as possible". This requires the definition of a criterion to specify what "as well as possible" means. A usual answer consists in considering the *median relation* defined as follows (see for instance Barthélemy and Monjardet 1981; Hudry et al. 2009 or Hudry and Monjardet 2010 for surveys and references on the median procedure; for a larger approach of the field of computational social choice, see Brandt et al. 2013 for instance). First, we consider the symmetric difference distance $\delta(R_i, R)$ between any relation R defined on X and any binary relation R_i of Π ($1 \le i \le v$). This quantity $\delta(R_i, R)$ measures the number of disagreements between R_i and R (see Sect. 2 for the expression of $\delta(R_i, R)$). Then we define the (usual) *remoteness* $\rho_1(\Pi, R)$ between Π and a relation R belonging to $\mathcal{C}$ as the sum of the distances $\delta(R_i, R)$ over i (see Barthélemy and Monjardet 1981):

$$\rho_1(\Pi, R) = \sum_{i=1}^{v} \delta(R_i, R).$$

Thus $\rho_1(\Pi, R)$ measures the total number of disagreements between R and the relations of Π, and $\rho_1(\Pi, R)/v$ measures the (arithmetical) mean number of disagreements between R and the preferences of Π.

A $\mathcal{C}$*-median relation* of Π is any relation R^* belonging to $\mathcal{C}$ which minimizes the remoteness ρ_1 with respect to Π over $\mathcal{C}$:

$$\rho_1\left(\Pi, R^*\right) = \min_{R \in \mathcal{C}} \rho_1(\Pi, R).$$

In this respect, the median relation can be considered as a relation fulfilling some structural properties (the ones defining $\mathcal{C}$) and minimizing the average dissatisfaction for the arithmetical mean.

From the point of view of the theory of NP-completeness (see Garey and Johnson 1979 for a reference on this subject), the complexity of the computation of a $\mathcal{C}$-median relation depends of course on the properties required from the median, i.e. on $\mathcal{C}$ (and, in some extent, on the nature of the preferences R_i), but this problem is generally NP-hard (for the definition of NP-hardness based on polynomial-time Turing reductions, see Garey and Johnson 1979) if we require from R^* properties usually considered as desirable, like transitivity (see Alon 2006; Bartholdi et al. 1989; Charbit et al. 2007; Charon and Hudry 2010; Conitzer 2006; Dwork et al. 2001; Hemaspaandra et al. 2005; Hudry 1989, 2008, 2010, 2012, 2013; Wakabayashi 1986, 1998 for complexity results dealing with the computation of $\mathcal{C}$-median relations for several sets $\mathcal{C}$; see also Hudry 2009a, 2009b for surveys on complexity results dealing with other voting procedures or with tournament solutions). Notice that the minimum value of v for which the problem is NP-hard is not always known and is also studied. For instance, for the aggregation of linear orders into a median linear order or a median complete preorder with respect to ρ_1 (Condorcet-Kemeny's problems, Condorcet 1785;

Kemeny 1959), it is known that the minimum even value of v is equal to 4 while the minimum odd value is not known (see Charon and Hudry 2010; Dwork et al. 2001; Hudry 2008, 2012). Similarly, for a profile of v tournaments (see Sect. 2 for the definition of this structure) and still with respect to ρ_1, the computation of a median linear order (this problem is also known as Slater's problem for $v = 1$, see Slater 1961; see Charon and Hudry 2010 for other names and equivalent statements) or of a median complete preorder is an NP-hard problem for any given value of v greater than or equal to 1 (see Alon 2006; Charbit et al. 2007; Conitzer 2006; Hudry 2010, 2012).

Then, we may wonder what happens, from the complexity point of view, if we adopt another definition for the remoteness. For instance, what happens if we try to minimize the maximum dissatisfaction? or if we consider another type of mean, for instance the geometric mean? The aim of this paper is to study this issue for several types of remoteness.

The paper is organized as follows: Sect. 2 provides the basic definitions and states the problem; the complexity results can be found in Sect. 3; the conclusion (Sect. 4) summarizes these results and states some open problems related to these issues.

2 Definitions and notation

Let $X = \{1, 2, \ldots, n\}$ be a finite set with n elements. A binary relation R defined on X is a subset of the Cartesian product $X \times X$. If (x, y) belongs to R, then we write xRy; otherwise, we write $x\bar{R}y$. Given a binary relation R, we may define an asymmetric relation R^a (called the *asymmetric part* of R) by: $xR^ay \Leftrightarrow (xRy \text{ and } y\bar{R}x)$.

Basic properties that R may fulfill are:

* *reflexivity*: R is reflexive if, for any $x \in X$, we have xRx;
* *irreflexivity*: R is irreflexive if, for any $x \in X$, we have $x\bar{R}x$;
* *antisymmetry*: R is antisymmetric if, for any $(x, y) \in X^2$ with $x \neq y$, we have the implication $xRy \Rightarrow y\bar{R}x$;
* *completeness*: R is complete if, for any $(x, y) \in X^2$ with $x \neq y$, we have xRy or yRx;
* *transitivity*: R is transitive if, for any $(x, y, z) \in X^3$, we have the implication $(xRy$ and $yRz) \Rightarrow xRz$.

With respect to the complexity status, reflexivity or irreflexivity usually do not matter: for the usual remoteness ρ_1, the complexity remains the same with or without this property (see Hudry 2008). As the results of Sect. 3 are based on transformations involving ρ_1, they will not depend on requirements dealing with reflexivity or irreflexivity for the different types of remoteness studied in Sect. 3.

From these basic properties, we may define partially ordered structures (see Bouyssou et al. 2006 or Caspard et al. 2012). The structure of linear orders is surely the most studied one, but complete preorders, and by the way the asymmetric part of complete preorders, appear quite often, for instance in Arrow's theorem (Arrow 1951) and in Kemeny's problem (Kemeny 1959). As the names used to denote the usual structures are not always the same in the literature, we specify them below. For reflexivity or irreflexivity properties, we do not specify anything: once again, it does not matter for the results of Sect. 3.

* *Tournament*: a tournament is a relation which is antisymmetric and complete; $\mathcal{T}$ will denote the set of tournaments.
* *Linear order*: a linear order is a relation which is antisymmetric, complete and transitive, i.e. also a transitive tournament; $\mathcal{L}$ will denote the set of linear orders.

* *Complete preorder*: a complete preorder is a relation which is complete and transitive; $\mathcal{P}$ will denote the set of complete preorders.
* *Weak order*: a relation R is a weak order if there exists a complete preorder of which the asymmetric part is R; $\mathcal{W}$ will denote the set of weak orders.

Moreover, $\mathcal{C}o$, $\mathcal{A}$ and $\mathcal{R}$ will respectively denote the set of complete relations, the set of antisymmetric relations and the set of all the binary relations defined on X. We say that $(x_1, x_2, \dots, x_k)$ is a *directed cycle* of a relation R if we have $x_1 R x_2, x_2 R x_3, \dots, x_{k-1} R x_k$ and $x_k R x_1$, where k is some appropriate integer with $k \geq 2$ and where $x_1, x_2, \dots, x_k$ are pairwise distinct elements of X. A relation is said to be *acyclic* if it contains no directed cycle; $\mathcal{A}c$ will denote the set of acyclic relations. Note the following inclusions:

- $\mathcal{L} \subset \mathcal{P} \subset \mathcal{R}$;
- $\mathcal{L} \subset \mathcal{W} \subset \mathcal{A}c \subset \mathcal{A} \subset \mathcal{R}$;
- $\mathcal{L} = \mathcal{C}o \cap \mathcal{A}c \subset \mathcal{T} = \mathcal{C}o \cap \mathcal{A} \subset \mathcal{R}$.

To define median relations, we use the *symmetric difference distance* δ between two relations R and R' both defined on X. This distance is defined by:

$$\delta(R, R') = |R \Delta R'|,$$

where Δ denotes the usual *symmetric difference* between sets. This distance, which is also sometimes called *Kemeny's distance*, *Dodgson's distance*, *Kendall's tau distance*, *swap distance*, *bubble-sort distance*..., has good axiomatic properties (see Barthélemy 1979 and Barthélemy and Monjardet 1981) and measures the number of disagreements between R and R':

$$\delta(R, R') = \left|\left\{(x, y) \in X^2 \colon [x R y \text{ and } x \bar{R}' y] \text{ or } [x \bar{R} y \text{ and } x R' y]\right\}\right|.$$

The usual *remoteness* (Barthélemy and Monjardet 1981) $\rho_1(\Pi, R)$ between a profile $\Pi = (R_1, R_2, \dots, R_v)$ and a binary relation R is defined by:

$$\rho_1(\Pi, R) = \sum_{i=1}^{v} \delta(R_i, R).$$

So, the remoteness $\rho_1(\Pi, R)$ measures the total number of disagreements between Π and R. Given a prescribed set $\mathcal{C}$ of relations, a $\mathcal{C}$*-median relation*, or simply a *median relation* when there is no ambiguity, is a relation R^* belonging to $\mathcal{C}$ and minimizing ρ_1 over $\mathcal{C}$:

$$\rho_1(\Pi, R^*) = \min_{R \in \mathcal{C}} \rho_1(\Pi, R).$$

Sometimes, the computation of R^* can be done in polynomial time with respect to the size of the instance Π. It is the case for instance when $\mathcal{C}$ is $\mathcal{T}$ or $\mathcal{R}$ (see Hudry et al. 2009). In contrast, the computation of R^* for ρ_1 usually leads to NP-hard problems when R^* is required to own structural properties like transitivity or acyclicity. More precisely, when the relations R_i $(1 \leq i \leq v)$ of the profile Π belong to $\mathcal{R}$ or more generally to any set containing $\mathcal{L}$ as a subset, it is known that the computation of the $\mathcal{C}$-median is NP-hard when $\mathcal{C}$ is for instance equal to $\mathcal{A}c$, $\mathcal{L}$, $\mathcal{P}$, $\mathcal{W}$ (for more details, other results and references, see Alon 2006; Bartholdi et al. 1989; Charbit et al. 2007; Charon and Hudry 2010; Conitzer 2006; Dwork et al. 2001; Hemaspaandra et al. 2005; Hudry 1989, 2008, 2010, 2012; Wakabayashi 1986, 1998).

What happens if we consider another definition for the remoteness? For instance, instead of ρ_1, we may pay attention to the kinds of remoteness described below. As the expression *median relation* applies only when the considered remoteness is ρ_1, we will adopt the following notation. Given a remoteness ρ, a subset $\mathcal{D}$ (for *data*) of $\mathcal{R}$, a set $\mathcal{C}$ (for *central*) of partially ordered relations (which is also a subset of $\mathcal{R}$), a positive integer v, a profile Π of v binary relations belonging to $\mathcal{D}$, we say that R^* is *a* $(\rho, \mathcal{C})$*-central relation of* Π if R^* belongs to $\mathcal{C}$ and minimizes $\rho(\Pi, R)$ over $\mathcal{C}$:

$$\rho\big(\Pi, R^*\big) = \min_{R\in C} \rho(\Pi, R).$$

Thus the usual definition of the $\mathcal{C}$-median relation is the same as the definition of the $(\rho_1, \mathcal{C})$-central relation. When there is no ambiguity, we simply say that R^* is the $\mathcal{C}$*-central relation of* Π or the *central relation of* Π.

We now describe several types of remoteness that we may consider instead of ρ_1.

2.1 Maximum number of disagreements between Π and R

For the maximum number of disagreements between Π and R, the remoteness $\rho_{\max}$ is defined by:

$$\rho_{\max}(\Pi, R) = \max_{1\le i\le v} \delta(R_i, R).$$

Then a $(\rho_{\max}, \mathcal{C})$-central relation R^* minimizes the maximum dissatisfaction of the voters:

$$\rho_{\max}\big(\Pi, R^*\big) = \min_{R\in C} \rho_{\max}(\Pi, R) = \min_{R\in C} \max_{1\le i\le v} \delta(R_i, R).$$

2.2 Minimum number of disagreements between Π and R

Instead of minimizing the dissatisfaction of the most dissatisfied voter, we may try also to minimize the dissatisfaction of the least dissatisfied voter, or equivalently to maximize the satisfaction of the most satisfied voter. This leads to the remoteness $\rho_{\min}$ defined by:

$$\rho_{\min}(\Pi, R) = \min_{1\le i\le v} \delta(R_i, R).$$

Then a $(\rho_{\min}, \mathcal{C})$-central relation R^* is defined by:

$$\rho_{\min}\big(\Pi, R^*\big) = \min_{R\in C} \rho_{\min}(\Pi, R) = \min_{R\in C} \min_{1\le i\le v} \delta(R_i, R).$$

2.3 L^p-norms

The remoteness ρ_1 defined above can be related to the usual norm L^1. For any integer p with $p \ge 1$, we can generalize it by considering the norm L^p. We then obtain a remoteness ρ_p defined, for any profile Π and any relation R, by:

$$\rho_p(\Pi, R) = \left(\sum_{i=1}^{v} \delta(R_i, R)^p\right)^{1/p}.$$

A $(\rho_p, \mathcal{C})$-central relation R^* minimizes this new kind of remoteness:

$$\rho_p\big(\Pi, R^*\big) = \min_{R\in C} \rho_p(\Pi, R) = \min_{R\in C}\left(\sum_{i=1}^{v} \delta(R_i, R)^p\right)^{1/p}.$$

2.4 Geometric or harmonic means

As said above, ρ_1/v is the usual arithmetical mean. More generally, $\rho_p/(v^{1/p})$ defines a mean (the one associated with $p = 2$ is called the *quadratic mean*). Other kinds of means exist and we can consider the cases for which the remoteness is provided by the geometric mean ρ_g or the harmonic mean ρ_h.

The geometric mean ρ_g is defined by:

$$\rho_g(\Pi, R) = \left(\prod_{i=1}^{v} \delta(R_i, R)\right)^{1/v}.$$

If none of the v considered quantities $\delta(R_i, R)$ is equal to 0 for $1 \le i \le v$ (otherwise we set $\rho_h(\Pi, R) = 0$ because it will be convenient in the following), the harmonic mean ρ_h is defined by:

$$\frac{v}{\rho_h(\Pi, R)} = \sum_{i=1}^{v} \frac{1}{\delta(R_i, R)}.$$

As before, a $\mathcal{C}$-central relation R^* with respect to ρ_g or ρ_h minimizes these means:

$$\rho_g\big(\Pi, R^*\big) = \min_{R \in \mathcal{C}} \rho_g(\Pi, R) = \min_{R \in \mathcal{C}} \left(\prod_{i=1}^{v} \delta(R_i, R)\right)^{1/v}$$

and

$$\rho_h\big(\Pi, R^*\big) = \min_{R \in \mathcal{C}} \rho_h(\Pi, R).$$

2.5 Weighted remoteness

We may also consider a remoteness in which the voters do not play symmetric roles. For instance, let $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ be a v-tuple of numbers with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_v \ge 0$ and $\alpha_1 > 0$ (in the following, it is not always necessary to assume that these numbers are non-negative, but assuming this non-negativity will avoid being more specific when necessary; from a practical point of view, this assumption is not restrictive). We then define the "weighted" remoteness $\rho_{1,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$ by:

$$\rho_{1,(\alpha_1,\alpha_2,\ldots,\alpha_v)}(\Pi, R) = \sum_{i=1}^{v} \alpha_i \delta(R_i, R).$$

For $(\alpha_1, \alpha_2, \ldots, \alpha_v) = (1, 1, \ldots, 1)$, we get the usual remoteness ρ_1 back. But we can represent other situations in such a way. For instance, the dictatorship (only one voter, the dictator, imposes his or her preferences) can be obtained by choosing $(\alpha_1, \alpha_2, \ldots, \alpha_v) = (1, 0, \ldots, 0)$. The persecution for which the preferences of one voter are systematically not taken into account can be associated with the v-tuple $(1, \ldots, 1, 0)$. More generally, for a given k between 1 and v, a k-coalition for which only k voters are taken into account, can be associated with the v-tuple $(1, \ldots, 1, 0, \ldots, 0)$ where 1 is repeated k times. We may still imagine other situations. For instance, in some associations, the president of the association benefits from a weight more important than the other members; this can be expressed by the v-tuple $(\alpha_1, 1, \ldots, 1)$ with $\alpha_1 > 1$. Or we can also represent a kind of hierarchy between the voters by an v-tuple like $(v, v-1, v-2, \ldots, 2, 1)$. And so on …

Of course, in all these cases, the $(\rho_{1,(\alpha_1,\alpha_2,\dots,\alpha_v)}, \mathcal{C})$-central relation R^* is still defined as above:

$$\rho_{1,(\alpha_1,\alpha_2,\dots,\alpha_v)}\big(\Pi, R^*\big) = \min_{R\in C} \rho_{1,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = \min_{R\in C} \sum_{i=1}^{v} \alpha_i \delta(R_i, R).$$

Following the same idea, we may also generalize the types of remoteness defined in Sects. 2.1 to 2.4 by their weighted versions. We thus obtain:

- $\rho_{\max,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = \max_{1\le i\le v} \alpha_i \delta(R_i, R)$,
- $\rho_{\min,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = \min_{1\le i\le v} \alpha_i \delta(R_i, R)$,
- $\rho_{p,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = (\sum_{i=1}^{v} \alpha_i \delta(R_i, R)^p)^{1/p}$ for any integer $p \ge 1$,
- $\rho_{g,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = (\prod_{i=1}^{v} \delta(R_i, R)^{\alpha_i})^{1/\sum_{i=1}^{v} \alpha_i}$,
- and $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}$ defined by $\frac{\sum_{i=1}^{v} \frac{1}{\alpha_i}}{\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi,R)} = \sum_{i=1}^{v} \frac{1}{\alpha_i \delta(R_i,R)}$

if none of the quantities $\alpha_i \delta(R_i, R)$ is equal to 0 for $1 \le i \le v$ (such an assumption involves the inequality $\alpha_i > 0$ for any i); otherwise we set $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi, R) = 0$ as before.

For each type of remoteness, we recover the unweighted remoteness when the v-tuple $(\alpha_1, \alpha_2, \dots, \alpha_v)$ is equal to $(1, 1, \dots, 1)$.

2.6 The studied problems

The decision problem associated with the computation of a $(\rho, \mathcal{C})$-central relation and that we are going to study in Sect. 3 is defined as follows:

Definition 1 Let v be any integer with $v \ge 1$, ρ a remoteness, and $\mathcal{D}$ and $\mathcal{C}$ two subsets of $\mathcal{R}$. Problem $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is defined by:

Name: $D_{v,\rho,\mathcal{D},\mathcal{C}}$ (aggregation of v binary relations belonging to $\mathcal{D}$ into a $(\rho, \mathcal{C})$-central relation);
Instance: a set X of n elements, a profile Π of v relations defined on X and all belonging to $\mathcal{D}$; an integer k;
Question: does there exist a relation R defined on X and belonging to $\mathcal{C}$ with $\rho(\Pi, R) \le k$?

In particular, $D_{v,\rho_1,\mathcal{D},\mathcal{C}}$ is the decision problem associated with the computation of a $\mathcal{C}$-central relation for the usual remoteness ρ_1, i.e. with the computation of a $\mathcal{C}$-median relation. It is known that $D_{1,\rho_1,\mathcal{T},\mathcal{L}}$ (also known as Slater's problem, see Slater 1961), $D_{1,\rho_1,\mathcal{T},\mathcal{A}c}$ (also known as the Feedback Arc Set problem for tournaments, see Charon and Hudry 2010 for equivalent definitions and references), $D_{1,\rho_1,\mathcal{T},\mathcal{P}}$ and $D_{1,\rho_1,\mathcal{T},\mathcal{W}}$ are NP-complete problems (see Alon 2006; Charbit et al. 2007; Conitzer 2006; Hudry 2010, 2012).

3 Complexity results

We are going to study the complexity of the computation of a $(\rho, \mathcal{C})$-central relation for different sets of $\mathcal{C}$ and for different kinds of remoteness ρ. Theorem 1 below can be applied to many kinds of remoteness ρ, more precisely when ρ fulfills the property (P) that we define now:

Definition 2 Let v be any given integer with $v \ge 1$. Let $\mathbf{N}$ denote the set of non-negative integers. We say that ρ fulfills property (P) if there exists an increasing function φ_v (i.e.

such that we have $x < y \Leftrightarrow \varphi_v(x) < \varphi_v(y)$) from $\mathbf{N}$ to $\mathbf{N}$ such that, for any profile $\Pi = (R_1, R_1, \dots, R_1)$ containing v times a same preference R_1, we have, for any relation R:

$$\rho(\Pi, R) = \varphi_v\big(\rho_1(\Pi, R)\big).$$

Moreover, we assume that, for any integer k, $\varphi_v(k)$ can be computed in polynomial time with respect to the size of k.

In other words, when all the voters share the same opinion, the remoteness ρ is the same as the remoteness ρ_1 up to an increasing transformation, and this transformation can be computed in polynomial time.

When ρ fulfills property (P), the complexity of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ can be related to the one of $D_{1,\rho_1,\mathcal{D},\mathcal{C}}$, as stated by the next theorem.

Theorem 1 *Let v be any given integer with $v \geq 1$, ρ be any remoteness fulfilling property* (P), *and $\mathcal{D}$ and $\mathcal{C}$ be subsets of $\mathcal{R}$. Then $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is NP-complete as soon as we have*:

1. *$D_{v,\rho,\mathcal{D},\mathcal{C}}$ belongs to NP*;
2. *$D_{1,\rho_1,\mathcal{D},\mathcal{C}}$ is NP-complete.*

Proof We are going to transform $D_{1,\rho_1,\mathcal{D},\mathcal{C}}$ to $D_{v,\rho,\mathcal{D},\mathcal{C}}$ in polynomial time. For this, consider any instance I_1 of $D_{1,\rho_1,\mathcal{D},\mathcal{C}}$. Such an instance can be described by a set X, a unique relation R_1 defined on X, belonging to $\mathcal{D}$ and constituting a profile $\Pi_1 = (R_1)$, and by an integer k_1: $I_1 = (X, \Pi_1 = (R_1), k_1)$. The instance $I = (X, \Pi, k)$ of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ that we build from I_1 contains the same set X and the profile $\Pi = (R_1, R_1, \dots, R_1)$ containing v times the relation R_1. Moreover, as ρ is assumed to fulfill (P), there exists an increasing function φ_v such that we have, for any relation R: $\rho(\Pi, R) = \varphi_v(\rho_1(\Pi, R))$. Then the constant k involved in the instance I of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is defined by $k = \varphi_v(vk_1)$.

This transformation is polynomial, since v is fixed and thanks to the assumption about the polynomiality of φ_v.

Moreover, it keeps the answer. Indeed, assume that the instance I_1 of $D_{1,\rho_1,\mathcal{D},\mathcal{C}}$ admits the answer "yes". Then there exists a relation R belonging to $\mathcal{C}$ with $\rho_1(\Pi_1, R) \leq k_1$. By property (P), we have: $\rho(\Pi, R) = \varphi_v(\rho_1(\Pi, R))$. Since all the v relations of Π are the same, namely R_1, we have: $\rho_1(\Pi, R) = v\delta(R_1, R) = v\rho_1(\Pi_1, R)$. Hence: $\rho(\Pi, R) = \varphi_v(v\rho_1(\Pi_1, R))$. As φ_v is increasing, we obtain: $\rho(\Pi, R) \leq \varphi_v(vk_1) = k$. Thus I admits the answer "yes". Conversely, assume that the instance I of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ admits the answer "yes". Then there exists a relation R belonging to $\mathcal{C}$ with $\rho(\Pi, R) \leq k$. Because Π contains a fixed relation repeated v times and still because of property (P), we obtain $\rho(\Pi, R) = \varphi_v(\rho_1(\Pi, R))$. The inequality $\rho(\Pi, R) \leq k = \varphi_v(vk_1)$ becomes $\varphi_v(\rho_1(\Pi, R)) \leq \varphi_v(vk_1)$. Once again because φ_v is increasing, we deduce from the previous inequality: $\rho_1(\Pi, R) \leq vk_1$. This inequality and the equalities $\rho_1(\Pi, R) = v\delta(R_1, R) = v\rho_1(\Pi_1, R)$ involve $\rho_1(\Pi_1, R) \leq k_1$: then I_1 admits the answer "yes".

So, if in addition $D_{v,\rho,\mathcal{D},\mathcal{C}}$ belongs to NP and $D_{1,\rho 1,\mathcal{D},\mathcal{C}}$ is NP-complete, the conclusion is that $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is NP-complete. □

In the next theorems, the individual preferences R_i $(1 \leq i \leq v)$ of the considered profiles are assumed to belong to a set $\mathcal{D}$ containing at least $\mathcal{T}$. For instance, because of the relation $\mathcal{T} = \mathcal{C}o \cap \mathcal{A} \subset \mathcal{R}$, the relations R_i $(1 \leq i \leq v)$ can be assumed to be tournaments, complete relations, antisymmetric relations, or also binary relations without noticeable structural properties, or still a mixture of these kinds of relations.

The belonging of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ to NP is usually easy to prove. It is the case for the different types of remoteness of Sect. 2 and for different sets $\mathcal{C}$, as specified by Lemma 2.

Lemma 2 *For any integer v with $v \geq 1$, any v-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ of integers with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_v \geq 0$ and $\alpha_1 > 0$, any integer $p \geq 1$, any remoteness ρ belonging to $\{\rho_{\max,(\alpha_1,\alpha_2,\ldots,\alpha_v)}, \rho_{\min,(\alpha 1,\alpha_2,\ldots,\alpha_v)}, \rho_{p,(\alpha_1,\alpha_2,\ldots,\alpha_v)}, \rho_{g,(\alpha_1,\alpha_2,\ldots,\alpha_v)}, \rho_{h,(\alpha_1,\alpha_2,\ldots,\alpha_v)}\}$, any set $\mathcal{D}$ containing $\mathcal{T}$ and any set $\mathcal{C}$ belonging to $\{\mathcal{L}, \mathcal{P}, \mathcal{W}, \mathcal{A}c\}$, $D_{v,\rho,\mathcal{D},\mathcal{C}}$ belongs to NP.*

Proof We do not detail all the steps of the proof of Lemma 2, what would be tedious while the proof is not difficult. We just give the main features of the proof.

Let $I = (X, \Pi, k)$ denote any instance of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ admitting the answer "yes".

First, note that any relation R defined on X can be encoded with n^2 bits, with $n = |X|$: for each ordered pair (x, y) of elements of X, a 1 will code the fact that x and y are in relation with respect to R (i.e. xRy), while a 0 will code the contrary (i.e. $x\bar{R}y$). Moreover, if R is a tournament, the definition of R requires the specification of at least $n(n-1)/2$ bits: one bit for each pair $\{x, y\}$ of elements of X. As $\mathcal{D}$ contains $\mathcal{T}$ as a subset, there is no qualitatively more compact representation of the preferences belonging to Π than the previous one, with a size about n^2 for each preference belonging to Π. Thus the size of I is in $\theta(\log_2 n + vn^2 + \log_2 k)$.

Imagine now that we guess an element R belonging to $\mathcal{C}$ providing the answer "yes" to the instance I. We want to check, in polynomial time with respect to the size of I, that:

- R does belong to $\mathcal{C}$;
- R does provide the answer "yes".

Checking that R is a linear order, or a complete preorder, or the asymmetric part of a complete preorder, or an acyclic relation defined on X can easily be done in $O(n^2)$, i.e. in polynomial time with respect to the size of I, for instance by considering the strongly connected components of the directed graph induced by R, which can be computed in $O(n^2)$, (see Cormen et al. 1990).

Then, given R, the computation of $\rho(\Pi, R)$ can also be done in polynomial time with respect to the size of I for any remoteness ρ considered here. Indeed, the computation of $\delta(R_i, R)$ for any relation R_i $(1 \leq i \leq v)$ of the profile Π can be done in $O(n^2)$. Then, the computation of $\rho(\Pi, R)$ can be done in $O(vn^2)$ (or even in $O(n^2)$, since v is fixed), which can be once again upper-bounded by a polynomial in the size of I. Last, we just have to compare $\rho(\Pi, R)$ with k, what can be done in $O(1)$.

As all these steps can be done in polynomial time with respect to the size of I, $D_{v,\rho,\mathcal{D},\mathcal{C}}$ belongs to NP under the hypotheses of Lemma 2. □

Thanks to Theorem 1 and to complexity results dealing with ρ_1, we can now obtain new results about the other kinds of remoteness depicted in Sect. 2, since they fulfill property (P), as shown by Lemma 3.

Lemma 3

- *For any integer v with $v \geq 1$, any v-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ of numbers with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_v \geq 0$ and $\alpha_1 > 0$, any integer $p \geq 1$, $\rho_{\max,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$, $\rho_{p,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$ and $\rho_{g,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$ fulfill property* (P).
- *For any integer v with $v \geq 1$, any v-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ of numbers with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_v > 0$, $\rho_{\min,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$, $\rho_{h,(\alpha_1,\alpha_2,\ldots,\alpha_v)}$ fulfill property* (P).

Proof Let Π_0 denote any profile $(R_0, R_0, \dots, R_0)$ in which a same binary relation R_0 is repeated v times. Then we have, for any binary relation R: $\rho_1(\Pi_0, R) = v\delta(R_0, R)$ or, equivalently: $\delta(R_0, R) = \rho_1(\Pi_0, R)/v$.

It is not difficult to check that the following functions φ_v satisfy the requirements of Definition 2.

- For $\rho_{\max,(\alpha_1,\alpha_2,\dots,\alpha_v)}$: $\varphi_v(x) = \alpha_1 x/v$; indeed, we have:

$$\rho_{\max,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R) = \max_{1\le i\le v} \alpha_i\delta(R_0, R) = \alpha_1\rho_1(\Pi_0, R)/v.$$

- For $\rho_{\min,(\alpha_1,\alpha_2,\dots,\alpha_v)}$: $\varphi_v(x) = \alpha_v x/v$; indeed, we have:

$$\rho_{\min,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R) = \min_{1\le i\le v} \alpha_i\delta(R_0, R) = \alpha_v\rho_1(\Pi_0, R)/v.$$

- For $\rho_{p,(\alpha_1,\alpha_2,\dots,\alpha_v)}$: $\varphi_v(x) = (\sum_{i=1}^{v} \alpha_i)^{1/p} x/v$; indeed, we have:

$$\rho_{p,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R) = \left(\sum_{i=1}^{v} \alpha_i\delta(R_0, R)^p\right)^{1/p} = \left(\sum_{i=1}^{v} \alpha_i\right)^{1/p} \delta(R_0, R).$$

- For $\rho_{g,(\alpha_1,\alpha_2,\dots,\alpha_v)}$: $\varphi_v(x) = x/v$; indeed, we have:

$$\rho_{g,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R) = \left(\prod_{i=1}^{v} \big(\delta(R_0, R)\big)^{\alpha_i}\right)^{1/\sum_{i=1}^{v}\alpha_i} = \delta(R_0, R) = \rho_1(\Pi_0, R)/v.$$

- For $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}$: $\varphi_v(x) = x/v$; indeed, we have:
  - if $\delta(R_0, R)$ is equal to 0, then $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R)$ is set to 0, and $\rho_1(\Pi_0, R)$ is also equal to 0;
  - otherwise, the terms $\alpha_i\delta(R_0, R)$ are not equal to 0 for $1 \le i \le v$; then we have:

$$\frac{\sum_{i=1}^{v} \frac{1}{\alpha_i}}{\rho_{h},_{(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R)} = \sum_{i=1}^{v} \frac{1}{\alpha_i\delta(R_0, R)},$$

  i.e. $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R) = \delta(R_0, R) = \rho_1(\Pi_0, R)/v$.

In both cases, $\rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}(\Pi_0, R)$ is equal to $\varphi_v(\rho_1(\Pi_0, R))$ with $\varphi_v(x) = x/v$. □

Theorem 4 specifies some complexity results dealing with the different kinds of remoteness described in Sect. 2 for any set $\mathcal{D}$ containing $\mathcal{T}$, what includes the cases $\mathcal{D} = \mathcal{R}$, $\mathcal{D} = \mathcal{A}$, $\mathcal{D} = \mathcal{C}o$.

Theorem 4

- *For any integer v with $v \ge 1$, any v-tuple $(\alpha_1, \alpha_2, \dots, \alpha_v)$ of integers with $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_v \ge 0$ and $\alpha_1 > 0$, any integer $p \ge 1$, any remoteness ρ belonging to $\{\rho_{\max,(\alpha_1,\alpha_2,\dots,\alpha_v)}, \rho_{p,(\alpha_1,\alpha_2,\dots,\alpha_v)}, \rho_{g,(\alpha_1,\alpha_2,\dots,\alpha_v)}\}$, and any set $\mathcal{D}$ containing $\mathcal{T}$, $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$ are NP-complete problems.*
- *For any integer v with $v \ge 1$, any v-tuple $(\alpha_1, \alpha_2, \dots, \alpha_v)$ of integers with $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_v > 0$, any remoteness ρ belonging to $\{\rho_{\min,(\alpha_1,\alpha_2,\dots,\alpha_v)}, \rho_{h,(\alpha_1,\alpha_2,\dots,\alpha_v)}\}$, and any set $\mathcal{D}$ containing $\mathcal{T}$, $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$ are NP-complete problems.*

Proof By Lemma 2, we know that $D_{v,\rho,\mathcal{T},\mathcal{L}}$, $D_{v,\rho,\mathcal{T},\mathcal{A}c}$, $D_{v,\rho,\mathcal{T},\mathcal{P}}$ and $D_{v,\rho,\mathcal{T},\mathcal{W}}$ belong to NP. Moreover, it is known that $D_{1,\rho_1,\mathcal{T},\mathcal{L}}$ (i.e. Slater problem) and $D_{1,\rho_1,\mathcal{T},\mathcal{A}c}$ (i.e. the Feedback Arc Set problem for tournaments) are NP-complete (see Alon 2006; Charbit et al. 2007;

Conitzer 2006; Hudry 2010), as well as $D_{1,\rho_1,\mathcal{T},\mathcal{P}}$ and $D_{1,\rho_1,\mathcal{T},\mathcal{W}}$ (see Hudry 2012). Since, by Lemma 3, ρ fulfills property (P), we may apply Theorem 1. Hence the results of Theorem 4 for $\mathcal{D} = \mathcal{T}$.

Now, consider any set $\mathcal{D}$ containing $\mathcal{T}$. By Lemma 2, we know that $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$ belong to NP. Then, for $\mathcal{C} \in \{\mathcal{L}, \mathcal{A}c, \mathcal{P}, \mathcal{W}\}$, let I be any instance of $D_{v,\rho,\mathcal{T},\mathcal{C}}$ and consider it as an instance of $D_{v,\rho,\mathcal{D},\mathcal{C}}$ (this is possible since $\mathcal{D}$ is assumed to contain $\mathcal{T}$). This transformation (the identity!) is obviously polynomial and keeps the answer "yes" or "no". Hence the NP-completeness of the problems $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$. □

Remarks

1. It is because reflexivity or irreflexivity do not matter (see Hudry 2008) for problems $D_{1,\rho_1,\mathcal{T},\mathcal{L}}$, $D_{1,\rho_1,\mathcal{T},\mathcal{A}c}$, $D_{1,\rho_1,\mathcal{T},\mathcal{P}}$ and $D_{1,\rho_1,\mathcal{T},\mathcal{W}}$ that they do not matter for problems $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$ either.
2. Quite obviously, problems $D_{v,\rho,\mathcal{D},\mathcal{L}}$, $D_{v,\rho,\mathcal{D},\mathcal{A}c}$, $D_{v,\rho,\mathcal{D},\mathcal{P}}$ and $D_{v,\rho,\mathcal{D},\mathcal{W}}$ become polynomial for $\rho = \rho_{\min,(\alpha_1,\alpha_2,\ldots,\alpha_{v-1},0)}$ or $\rho = \rho_{h,(\alpha_1,\alpha_2,\ldots,\alpha_{v-1},0)}$ since, in these cases, $\rho(\Pi, R)$ is always equal to 0 for any profile Π and any relation R. Hence the hypothesis $\alpha_v > 0$ (i.e., all the weights are positive) in the statement of Theorem 4 (otherwise, the functions φ_v would not be increasing).

In several places, we assume that v is fixed (it can be necessary to provide the polynomiality of some transformations). Theorem 5 states that the problems similar to $D_{v,\rho,\mathcal{D},\mathcal{C}}$ but without v fixed remain NP-complete as soon as $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is NP-complete for some values of v and for the types of remoteness of Sects. 2.1 to 2.4.

Theorem 5 *For any integer $p \geq 1$, any remoteness ρ belonging to $\{\rho_{\max}, \rho_{\min}, \rho_p, \rho_g, \rho_h\}$, any set $\mathcal{D}$ containing $\mathcal{T}$ and any set $\mathcal{C}$ belonging to $\{\mathcal{L}, \mathcal{A}c, \mathcal{P}, \mathcal{W}\}$, the following problem is NP-complete*:

Name: $D_{\rho,\mathcal{D},\mathcal{C}}$ (*aggregation of an unfixed number of binary relations belonging to $\mathcal{D}$ into a $(\rho, \mathcal{C})$-central relation*);
Instance: *a set X of n elements, an integer v, a profile Π of v relations defined on X and all belonging to $\mathcal{D}$; an integer k*;
Question: *does there exist a relation R defined on X and belonging to $\mathcal{C}$ with $\rho(\Pi, R) \leq k$?*

Proof As before, it is easy to show that the problems $D_{\rho,\mathcal{D},\mathcal{C}}$ belong to NP (details are left to the reader). Now, consider any instance of $D_{1,\rho,\mathcal{D},\mathcal{C}}$, which is an NP-complete problem by Theorem 4. Let I be any instance of $D_{1,\rho,\mathcal{D},\mathcal{C}}$ and consider it, with $v = 1$, as an instance of $D_{\rho,\mathcal{D},\mathcal{C}}$. Obviously, this transformation is polynomial and keeps the answer. Hence the results of Theorem 5. □

More generally, this result can be extended to any remoteness ρ such that $D_{\rho,\mathcal{D},\mathcal{C}}$ belongs to NP and such that there exists some constant c for which $D_{c,\rho,\mathcal{D},\mathcal{C}}$ is NP-complete. It is also possible to extend these results by introducing the weights $(\alpha_1, \alpha_2, \ldots, \alpha_v)$ into the instance when v depends also on the instance.

4 Conclusion

We may summarize the previous results as follows: the aggregation of v binary relations belonging to any set containing the set $\mathcal{T}$ of tournaments as a subset into a $(\rho, \mathcal{C})$-central relation is an NP-hard problem for any given value of v with $v \geq 1$, when $\mathcal{C}$ is the set $\mathcal{L}$ of linear orders, the set $\mathcal{A}c$ of acyclic relations, the set $\mathcal{P}$ of complete preorders or the set $\mathcal{W}$ of weak orders, for any remoteness ρ fulfilling property (P), which is the case of many types of remoteness, including the ones of Sect. 2. The similar problems in which v is not fixed but depends on the instance remain NP-hard in the same conditions (except the one on v, of course).

Two types of open problems related to these results can be studied.

The first one consists in considering other kinds of central relations: what happens for other sets $\mathcal{C}$, for instance if $\mathcal{C}$ stands for the set of partial orders? The answer to this question is not obvious. For instance, if $\mathcal{C}$ stands for the set of partial orders, the complexity of $D_{1,\rho_1,\mathcal{T},\mathcal{C}}$ is not known, i.e. for the usual remoteness ρ_1 when v is equal to 1 (some results are known for larger values of v or for other sets $\mathcal{C}$, see for instance Hudry 1989, 2008; Wakabayashi 1986, 1998).

The second one consists in considering other types of partially ordered sets for $\mathcal{D}$, i.e. other partially ordered structures to represent the individual preferences that we want to aggregate. For instance, what happens if all the individual preferences are linear orders (i.e. $\mathcal{D} = \mathcal{L}$)? Clearly, all the problems $D_{1,\rho,\mathcal{L},\mathcal{C}}$ for any remoteness ρ become polynomial for any set $\mathcal{C}$ with $\mathcal{L} \subseteq \mathcal{C}$ (what is quite often the case). More generally, all the problems $D_{1,\rho,\mathcal{D},\mathcal{C}}$ for any remoteness ρ are polynomial for sets $\mathcal{D}$ and $\mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{C}$: the unique relation of the considered profile provides a central relation. Thus, the approach applied in Sect. 3, based on NP-hardness results when v is equal to 1, is no longer possible to prove NP-hardness results. And, indeed, some problems are polynomial. For instance, for $\mathcal{D} = \mathcal{L}$ and $\mathcal{C} = \mathcal{L}$, $D_{1,\rho_1,\mathcal{L},\mathcal{L}}$ and $D_{2,\rho_1,\mathcal{L},\mathcal{L}}$ are polynomial (see Charon and Hudry 2010) while $D_{v,\rho_1,\mathcal{L},\mathcal{L}}$ is known, from Dwork et al. (2001), to be NP-complete for any even v with $v \geq 4$. Similarly, $D_{v,\rho_{\min},\mathcal{C},\mathcal{C}}$, $D_{v,\rho_g,\mathcal{C},\mathcal{C}}$ and $D_{v,\rho_h,\mathcal{C},\mathcal{C}}$ are obviously polynomial for any set $\mathcal{C}$ and any v (since any relation belonging to the profile that we want to aggregate provides an optimal solution, with a value equal to 0 for the remoteness). But what about the complexity for instance of $D_{v,\rho_{\max},\mathcal{L},\mathcal{L}}$ when v is greater than 1? We may conjecture that many problems $D_{v,\rho,\mathcal{D},\mathcal{C}}$ remain NP-complete, but not all of them, as we have just noticed. In this case, it would be also interesting to determine the minimum value of v from which $D_{v,\rho,\mathcal{D},\mathcal{C}}$ is NP-complete.

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