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Lexicographic decomposition of preferences

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Abstract

The *Ordinal Lexicographic Model*, based on the *lexicographic sum of relations*, provides an explanation to intransitivities in preferences as shifts in choice criteria. The lexicographic sum $R \oplus R'$ between two preferences R and R' on a same finite set X is defined as follows: x is preferred to y with respect to $R \oplus R'$ when x is preferred to y with respect to R, or there is a tie between x and y with respect to R but x is preferred to y with respect to R'. If a preference R can be written as $R = R_1 \oplus R_2 \oplus ... \oplus R_k$, $(R_1, R_2, ..., R_k)$ is said to be a *lexicographic decomposition* (LD) of R. It means that a first criterion, expressed by R_1, explains a part of R, a second criterion, expressed by R_2, explains a part of R which is not explained by R_1, and so on. Each relation R_i of the LD can be interpreted as a point of view, and the number k is the number of shifts in point of view in the decomposition. It is usually required from the relations of the LD to fulfil some structural properties, for instance to be partial orders. When a LD is possible, the usual question consists in computing, for any preference R, the minimum number $d(R)$, called the *lexicographic dimension of* R, of relations involves in the LD in order to explain R. The aim of this paper is to provide some properties of the lexicographic sum $\oplus$.

Keywords: Preferences, Intransitivities, Lexicographic Sum, Lexicographic Model, Lexicographic Decomposition, Lexicographic Dimension.

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1 Introduction

In the field of preferences analysis, when we pairwise compare alternatives in order to rank them, it often happens to observe intransitivities: if x, y and z denote three of these alternatives, x may be considered as better than y and y as better than z by a decider, while z is considered as better than x by the same decider. Different rules have been suggested and studied in order to explain these intransitivities. For instance, maximin and maximax rules [7], conjunctive and disjunctive rules [6], [8], the majority rule [11], [13], the weighted set of differences rule [10], the choice by greatest attractiveness rule [12], the lexicographic rule [9], the minimum difference lexicographic rule [12], the lexicographic semiorder rule [14], the sequential accumulation of differences rule [1], the moving basis heuristics [3], [4]. In the lexicographic rules, alternatives are assumed to be characterized by several attributes, each attribute allowing to rank the alternative in terms of importance. Then the alternative ranked first is the one with the largest value for the most important attribute. If these is a tie with respect to this attribute, then the decision is based on the second most important attribute, and so on.

The *Ordinal Lexicographic Model* (OLM) studied in [2] is based on the *lexicographic sum* defined as follows. Let R and R' be two binary relations defined on a given finite set X of n alternatives. The lexicographic sum of R and R' is the relation $R \oplus R'$ defined, for $(x, y) \in X^2$, by:

$$(x, y) \in (R \oplus R') \iff \{(x, y) \in R \text{ or } [(y, x) \notin R \text{ and } (x, y) \in R']\}.$$

In other words, for a context in which R and R' denote preferences, x is preferred to y with respect to $R \oplus R'$ when x is preferred to y with respect to the first relation, R, or there is a tie between x and y with respect to R but x is preferred to y with respect to the second relation, R'. With set theoretic notation, if R^r denotes the *reversed relation of* R defined by: $(x, y) \in R^r \iff (y, x) \in R$, then $R \oplus R'$ is equal to $R \cup (R' \setminus R^r)$.

Of course, we may consider the lexicographic sum of several relations in OLM. If a relation R can be written as $R = R_1 \oplus R_2 \oplus ... \oplus R_k$ (we shall see in Section 2 that the operation $\oplus$ is associative; so parentheses are not necessary) for some appropriate integer k and some appropriate relations R_i $(1 \leq i \leq k)$, we say that $(R_1, R_2, ..., R_k)$ is a lexicographic decomposition of R. Then it means that a first criterion, expressed by R_1, explains a part of R, a second criterion, expressed by R_2, explains a part of R which is not explained by R_1, a third criterion, expressed by R_3, explains a part of R which is neither explained by R_1 nor by R_2, and so on. Each relation R_i of the lexicographic decomposition is interpreted in [2] as a point of view, and the number k is the number of shifts in point of view. It is usually required from the relations of the lexicographic decomposition to fulfil some structural properties, for instance to be antisymmetric relations or partial orders (see below for the definitions of these structures); when the relations R_i $(1 \leq i \leq k)$ are assumed to be partial orders, then the decomposition is called an *Ordinal Lexicographic Decomposition* (OLD) of R in [2]. Similarly, the relation R that we want to decompose can also be assumed to fulfil properties, for instance to be a tournament. Then two questions arise in this context:

- Given some structural properties assumed to be fulfilled by the relation R to be decomposed and given some (other) structural properties required from the relations of the decomposition, is it possible to find such a lexicographic decomposition of R?
- If such a decomposition does exist, what is the minimum number of relations fulfilling the required properties involved in the lexicographic decomposition of R? In other words, what is the minimum number of shifts in point of view that we must involve to explain R with respect to the required properties? If the relations of the decomposition must belong to a set $\mathcal{S}$, this number is called the *lexicographic dimension of* R and will be noted $d_{\mathcal{S}}(R)$ in the following.

This paper provides some results dealing with these two issues. Section 2 specifies some basic algebraic properties. Section provides some possibility results dealing with the decomposition of antisymmetric relations into linear orders, partials orders or acyclic relations, and states some open problems.

2 Algebraic properties of lexicographic decomposition

A very basic property comes obviously from the definition of $\oplus$:

Lemma 2.1 *For any relations R and R', we have $R \subseteq R \oplus R' \subseteq R \cup R'$.*

In the following, we assume that the considered relations R are antisymmetric: $(x, y) \in R$ and $(y, x) \in R$ involve $x = y$. Let $\mathcal{A}(X)$ be the set of antisymmetric relations defined on X. Then we may easily characterize when $R \oplus R'$ is equal to R or to $R \cup R'$ (Proposition 2.6 will provide other characterizations, related to the communitativity of $\oplus$, of the equality $R \oplus R' = R \cup R'$):

Proposition 2.2
1. $R = R \oplus R' \iff R' \subseteq R \cup R^r$.
2. $R \oplus R' = R \cup R' \iff R^r \cap R' = \emptyset$.

Proof. 1. If $R = R \oplus R'$, then R' does not bring any new ordered pair. So it means that, if we have $(x, y) \in R'$, (x, y) already belongs to R or (y, x) prevents (x, y) from being in $R \oplus R'$ because (y, x) itself belongs to R or, equivalently, because (x, y) belongs to R^r. In both cases, it involves that R' is a subset of $R \cup R^r$. Conversely, if R' is a subset of $R \cup R^r$, then obviously R' cannot bring any new ordered pair to the ones of R and so R is equal to $R \oplus R'$.
2. As $R \oplus R'$ is equal to $R \cup (R' \setminus R^r)$, $R' \cap R^r = \emptyset$ obviously involves that $R \oplus R'$ is equal to $R \cup R'$. Conversely, assume that $R^r \cap R'$ is not the empty set. Let (x, y) belong to $R^r \cap R'$. As (x, y) belongs to R^r, then (y, x) belongs to R and (x, y) does not belong to R. Then (x, y) cannot belong to $R \oplus R'$ (since we have neither $(x, y) \in R$ nor $[(y, x) \notin R$ and $(x, y) \in R'])$ while it belongs to $R^r \cap R'$. So, in this case $R \oplus R'$ and $R \cup R'$ are distinct. □

The operation $\oplus$ defines a monoid on $\mathcal{A}(X)$ (in other words, $\oplus$ is associative and admits a neutral element) of which the neutral element is the empty relation $\emptyset$. This is specified by Theorem 2.3:

Theorem 2.3 *The operation $\oplus$ defines a monoid on $\mathcal{A}(X)$.*

Proof.

First, note that, if R and R' belong to $\mathcal{A}(X)$, then $R \oplus R'$ belongs to $\mathcal{A}(X)$ too. Indeed, let (x, y) belong to $R \oplus R'$. If (x, y) belongs to R, then (y, x) does not belong to R because R is antisymmetric; then the belonging of (y, x) to $R \oplus R'$ could only come from R'; but, even if (y, x) does belong to R', the belonging of (x, y) to R prevents (y, x) from belonging to $R \oplus R'$. If (x, y) does not belong to R, the belonging of (x, y) to $R \oplus R'$ requires simultaneously that (x, y) belongs to R' and that (y, x) does not belong to R; in this case, the belonging of (y, x) to $R \oplus R'$ would require the belonging of (y, x) to R', what is impossible because of the antisymmetry of R'. In both cases, if (x, y) belongs to $R \oplus R'$, then (y, x) does not belong to $R \oplus R'$: $R \oplus R'$ is thus antisymmetric.

Let us show now that $\oplus$ is associative. Let x and y be elements of X and let R, R' and R'' be three antisymmetric relations defined on X. Then we have the following equivalences:

$$(x, y) \in (R \oplus R') \oplus R'' \iff (x, y) \in R \oplus R' \text{ or } \{(y, x) \notin R \oplus R' \text{ and } (x, y) \in R''\}$$

$$\iff (x, y) \in R \text{ or } [(y, x) \notin R \text{ and } (x, y) \in R')]$$

$$\text{or } \{(y, x) \notin R \text{ and } [(x, y) \in R \text{ or } (y, x) \notin R'] \text{ and } (x, y) \in R''\}$$

$$\iff (x, y) \in R \text{ or } [(y, x) \notin R \text{ and } (x, y) \in R')]$$

$$\text{or } \{(y, x) \notin R \text{ and } (y, x) \notin R' \text{ and } (x, y) \in R''\},$$

because of the antisymmetry of R.
Similar computations for $(x, y) \in R \oplus (R' \oplus R'')$ give the same result. Hence the associativity.

To conclude, observe that $\emptyset$ is the neutral element. Indeed, we obviously have the following: $R \oplus \emptyset = R \cup (\emptyset \setminus R^r) = R$ and $\emptyset \oplus R = \emptyset \cup (R \setminus \emptyset) = R$.□

A consequence of the next proposition will be that the only relation R admitting an inverse for $\oplus$ (i.e. a relation R^i with $R \oplus R^i = R^i \oplus R = \emptyset$) is $\emptyset$.

Proposition 2.4 *The only solution of the equation $R \oplus R' = \emptyset$ is $R = R' = \emptyset$.*

Proof. By Lemma 2.1, we know that R is a subset of $R \oplus R'$; hence $R = \emptyset$. Then, as $\emptyset$ is the neutral element of $\oplus$, $R \oplus R' = R' = \emptyset$. □

It is easy to characterize the cases for which we have $R \oplus R' = R$ or $R \oplus R' = R'$. Such a characterization is provided by Proposition 2.5.

Proposition 2.5 *Let R and R' be two relations belonging to $\mathcal{A}(X)$. Then we have:*

1. *$R \oplus R' = R \iff R' \subseteq R \cup R^r$;*
2. *$R \oplus R' = R' \iff R \subseteq R'$.*

Proof. 1. Assume that R' is not a subset of $R \cup R^r$ and let (x, y) belong to $R' \setminus (R \cup R^r)$. Because (y, x) does not belong to R (since (x, y) does not belong to R^r), we have $(x, y) \in R \oplus R'$ while (x, y) is assumed not to be in R. So if R' is not a subset of $R \cup R^r$, then $R \oplus R'$ is not equal to R. Conversely, it is easy to check that, if R' is a subset of $R \cup R^r$, then we have $R \oplus R' = R$.

2. By Lemma 2.1, we know that R is a subset of $R \oplus R'$. Thus $R \oplus R' = R'$ yields $R \subseteq R'$. Conversely, assume that R is a subset of R'. By Lemma 2.1, we know that $R \oplus R'$ is a subset of $R \cup R'$, i.e. of R': $R \oplus R' \subseteq R'$. Let (x, y) be in R'; because of the antisymmetry of R', (y, x) does not belong to R' and thus does not belong to R. So we have $(y, x) \notin R$ and $(x, y) \in R'$, what involves $(x, y) \in R \oplus R'$: $R' \subseteq R \oplus R'$. Hence the equality between the two sets. □

In general, the operation $\oplus$ is not commutative. For instance, for $X = \{1, 2\}$, let R be $\{(1, 2)\}$ and R' be $\{(2, 1)\}$, then $R \oplus R' = R$ and $R' \oplus R = R'$. Proposition 2.6 characterizes the cases for which we have $R \oplus R' = R' \oplus R$.

Proposition 2.6 *We have the following equivalencies:* $R \oplus R' = R' \oplus R \iff R^r \cap R' = \emptyset \iff R \cap R'^r = \emptyset \iff R \oplus R' = R \cup R' \iff R' \oplus R = R \cup R'$.

Proof. First, observe the equality $R \cap R'^r = (R^r \cap R')^r$. So $R \cap R'^r$ is empty if and only if $R^r \cap R'$ is also empty. This and the second statement of Proposition 2.2 involve the equivalencies between statements 2 to 5.

Statements 4 and 5 obviously involve statement 1. So statement 2, which is equivalent to statement 4 and to statement 5, involves statement 1. Assume now that $R^r \cap R' \neq \emptyset$ and let (x, y) be in $R^r \cap R'$. Then, as (x, y) belongs to R', (x, y) belongs to $R' \oplus R$. But as (x, y) belongs to R^r, (y, x) belongs to R and (x, y) does not belong to R; then (x, y) cannot belong to $R \oplus R^r$ (we have neither $(x, y) \in R$ nor $[(y, x) \notin R$ and $(x, y) \in R']$). So, in this case, $R \oplus R'$ and $R' \oplus R$ are not equal. So statements 1 and 2 are equivalent and, finally, all these statements are equivalent. □

3 Possibility results and open problems

Rationality often requires transitivity, like in linear orders (i.e. antisymmetric, complete and transitive relations; for references on ordered structures, see for instance [5]). Anyway linear orders are usually too restrictive, as shown by the next proposition.

Proposition 3.1 *The only antisymmetric relations which admit a lexicographic decomposition into linear orders are the linear orders. In this case, the lexicographic dimension of any linear order is 1.*

Proof. Indeed, let R be an asymmetric relation admitting a lexicographic decomposition $R = O_1 \oplus O_2 \oplus ... \oplus O_k$. By Lemma 2.1, O_1 is a subset of R. As R is antisymmetric and O_1 is complete, R must be equal to O_1. Hence the statement of the proposition. □

Two less restrictive structures are the partial orders (i.e. antisymmetric and transitive relations) or the acyclic relations (relations A such that, for any $k \geq 2$ and any k-tuple $(i_1, i_2, ..., i_k)$, $(i_1, i_2) \in A, (i_2, i_3) \in A, ..., (i_{k-1}, i_k) \in A$ involve $(i_k, i_1) \notin A$; note that an acyclic relation is not necessarily transitive). Let $\mathcal{P}(X)$ and $\mathcal{AC}(X)$ denote the sets of partial orders defined on X and of acyclic relations defines on X respectively. Note the inclusion $\mathcal{P}(X) \subset \mathcal{AC}(X)$.

Let i and j be integers between 1 and n. Define the partial order O_{ij} by $(i, j) \in O_{ij}$ and, for $(x, y) \neq (i, j), (x, y) \notin O_{ij}$. Note the equality $O_{ij}^r = O_{ji}$; thus, for $(i, j) \neq (j', i')$, we have $O_{ij}^r \cap O_{i'j'} = \emptyset$. A consequence of this and of

Proposition 2.6 is that the lexicographic sum of partial orders of type O_{ij} is commutative and the order in which we sum these orders has no importance. These partial orders show that it is always possible to lexicographically decompose any antisymmetric relation (in fact, any binary relation) into partial orders or acyclic relations, as specified by the next theorem:

Theorem 3.2 *Any nonempty antisymmetric relation R can be lexicographically decomposed into at most $|R|$ partial orders.*

Proof. Indeed, R may be written as $R = \oplus\{O_{ij} \text{ for } (i,j) \in R\}$ (as noticed above, the order in way the sum is performed has no importance since R is antisymmetric). □

A consequence of Theorem 3.2 is that the lexicographic dimension of R into partial orders is at most $|R|$: $d_{\mathcal{P}(X)}(R) \leq |R|$. Thus the maximum value of $d_{\mathcal{P}(X)}$ over $\mathcal{A}$ is upper-bounded by $n(n-1)/2$. Another consequence is that the lexicographic dimension of R into acyclic relations is also always possible with at most $|R|$ acyclic relations. In fact, it is possible to be more specific, as shown by Theorem 3.3:

Theorem 3.3 *Any nonempty antisymmetric relation R can be lexicographically decomposed into acyclic relations. Moreover, if R is a linear order, then $d_{\mathcal{AC}(X)}(R) = 1$; otherwise, $d_{\mathcal{AC}(X)}(R) = 2$.*

Proof. We just must prove that, if R is not a linear order, then $d_{\mathcal{AC}(X)}(R) = 2$ (the other results are consequences of Proposition 3.1 and Theorem 3.2). Indeed, consider any linear order O defined on X. Consider the two relations A_1 and A_2 defined by $A_1 = R \cap O$ and $A_2 = R \cap O^r$. Then A_1 and A_2 obviously belong to $\mathcal{AC}(X)$ (since O is acyclic) and moreover R is equal to $A_1 \cup A_2$. As $A_1 \cap A_2^r = \emptyset$, we obtain, by Proposition 2.6, $R = A_1 \cup A_2$. Hence the result about the lexicographic dimension of R. □

Theorem 3.3 utterly characterizes the lexicographic decomposition of any antisymmetric relation into acyclic relations. But what about the decomposition into partial orders? We finish this section with two open problems.

Problem 3.4 *Given any antisymmetric relation R, what is the complexity of the computation of $d_{\mathcal{P}(X)}(R)$?*

Problem 3.5 *Given a finite set X, what is the maximum value of $d_{\mathcal{P}(X)}(R)$ for $R \in \mathcal{A}(X)$?*

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