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Cyclic colliding permutations

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Abstract

We study lower and upper bounds for the maximum size of a set of pairwise cyclic colliding permutations.

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1 Preliminaries

We say that two permutations $x, y \in S_n$ are *cyclic colliding* if and only if there exists an index $1 \leq i \leq n$ such that the images of i according to x and y differ by 1 modulo n.

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More generally we consider

$$T_m(n) = \max\{|C| : C \subseteq S_n, \forall \{x, y\} \in \binom{C}{2} \ \exists i \in [n] : \ |x_i - y_i| \equiv 1(mod\ m)\}.$$

We want to determine $T_m(n)$ and $T^*(n) = T_n(n)$ at least asymptotically. This is in analogy with a similar problem introduced by Körner and the second author in [3]: two permutations $x, y \in S_n$ are *colliding* if and only if there exists an index $1 \leq i \leq n$ such that the images of i by x and y differ by 1. The best known lower bound for $T(n) := T_{n+1}(n)$, that is the maximum size of a set of pairwise colliding permutations, can be found in [1].

We say that two permutations in $T_m(n)$ are *m-colliding*. This incompasses both the definitions of cyclic colliding permutations (when $m = n$) and colliding permutations (when $m > n$).

The following is obvious.

Proposition 1.1 *If m' divides m, then $T_{m'}(n) \geq T_m(n)$.*

We define the *parity pattern* (*pp*) of a permutation $x = (x_1, x_2, \ldots, x_n)$ by $pp(x) = (x_1[2], x_2[2], \ldots, x_n[2])$. For example, if $x = \mathbf{1} := (1, 2, \ldots, n)$ is the identical permutation, then $pp(x) = (1, 0, 1, 0, \ldots)$. Since the parity pattern of a permutation is balanced binary sequence, there are $\binom{n}{\lfloor n/2 \rfloor}$ possible parity patterns.

Setting xRy if and only if $pp(x) = pp(y)$ defines an equivalence relation, with each class associated to a parity pattern. Clearly, if m is even, two m-colliding permutations cannot have the same parity pattern, i.e. $x \to pp(x)$ restricted to a $m = 2m'$-colliding code is injective. Thus

Proposition 1.2 $T_{2m'}(n) \leq \binom{n}{\lfloor n/2 \rfloor}$.

When $m' = 1$, equality holds, since two permutations belonging to different classes will be 2-colliding:

Proposition 1.3 $T_2(n) = \binom{n}{\lfloor n/2 \rfloor}$.

We now focus on the case of cyclic collision ($m = n$).

2 Lower bounds

It is immediate to see that if two permutations are colliding, then they are cyclic colliding, that is, $T^*(n) \geq T(n)$. This can be improved:

Proposition 2.1 $T^*(n) \geq 2T(n-1)$.

Proof. Let $C \subseteq S_{2,\ldots,n}$ and $D \subseteq S_{1,\ldots,n-1}$ two sets of permutations of length $n-1$ pairwise colliding of maximum cardinality $T(n-1)$. The set of permutations of $[n]$

$$E := 1 \cdot C \ \cup \ n \cdot D$$

obtained by prefixing every permutation in C by 1 and every permutation in D by n is clearly pairwise cyclic colliding, and $|E| = 2T(n-1)$. $\square$

3 Upper bounds

We distinguish two cases, depending on the parity of n; the even case $n = m = 2m'$ follows from Proposition 1.2.

Proposition 3.1 $T^*(2m) \leq \binom{2m}{m}$.

Now we analyze the case when n is odd.

Proposition 3.2 $T^*(2m+1) \leq 3\binom{2m+1}{m}$.

Proof. Let σ be a permutation of $S_n, n = 2m+1$. The Hamming weight (i.e. the number of 1's) of $pp(\sigma)$ is $m+1$, thus there are $\binom{2m+1}{m}$ parity patterns. Let now C be a cyclic colliding code; observe that in this case the map pp is no longer injective as in the case of n even: however we want to prove that pp is at most 3–to–1 when restricted on C. Without loss of generality, let $z := (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ be the parity pattern of some codeword, say: $c^1 = (1, 3, 5, \ldots, 2m+1, 2, 4, \ldots, 2m)$. Let $D := pp^{-1}(z) = \{c^1, c^2, \ldots\}$ be the pre-image of z in C: we want to show that $|D| \leq 3$. Obviously the property of being cyclic colliding is inherited to subsets of a any cyclic colliding code (it is a pairwise condition holding for all pairs of the code); hence for D to be cyclic colliding, we must have: for $i \neq j$, c^i and c^j have the pair $\{1, 2m+1\}$ in some position (it is indeed the only way to be cyclic colliding and have the same parity pattern). Thus they never have a 1 nor a $2m+1$ in the same position.

Thus, without loss of generality, either:

$$\begin{array}{lcl} c^1 & = & (\ 1, \ 2m+1, \ *, \ *, \ \ldots, \ *) \\ c^2 & = & (\ 2m+1, \ 1, \ *, \ *, \ \ldots, \ *) \end{array}$$

and $|D| = 2$; or

$$\begin{array}{lcl} c^1 & = & (\ 1, \ *, \ 2m+1, \ *, \ *, \ \ldots, \ * \ ) \\ c^2 & = & (\ 2m+1, \ 1, \ *, \ *, \ *, \ \ldots, \ * \ ) \\ c^3 & = & (\ *, \ 2m+1, \ 1, \ *, \ *, \ \ldots, \ * \ ) \end{array}$$

and $|D| = 3$. □

Proposition 3.3

$$T^*(n) \leq nT(n-1).$$

Proof. We partition the permutations of a "code" C (that is, a family of permutations of n with the maximum cardinality with respect to the property of being pairwise cyclic colliding), according to the positions of the digit 1: let $C_j = \{x = (x_1, \dots, x_n) \in C : x_j = 1\}$, so that $C = C_1 \cup \ldots \cup C_n$, with the C_j's all disjont (possibly empty). Each C_j contains permutations that are pairwise colliding, where the digits $\{2, \dots, n\}$ are responsible for the collisions in C_j (since the cyclic collisions due to the digits 1 and n cannot appear in the C_j), so that $|C_j| \leq T(n-1)$. □

Corollary 3.4

$$2T(n-1) \leq T^*(n) \leq nT(n-1).$$

Remarks and questions

(i) In the case of cyclic collision, there is no proof of supermultiplicativity as for $T(n)$, namely : $T(n+m) \geq T(n)T(m)$; thus the determination of $T^*(n)$ cannot be seen as a "capacity" problem [1,3]. Setting $R_n := (1/n)\log_2 T_n$, we have by Fekete's lemma that R_n tends to a limit R as n goes to infinity. Thanks to the previous corollary, we get directly the convergency of the analogous quantity in the cyclic case; furthermore, $R* = R$ holds.

(ii) Can we prove/disprove that $T^*(n) \leq T^*(n+1)$?

(iii) Can we prove/disprove that $T^*(n) \leq T(n+1)$?

We know the values of $T(n)$ up to $n = 9$ (the cases of 8 and 9 were found independently by Adolfo Piperno and Brik [2], communicated by Adriano Garsia), and they are both of the form $\binom{n}{n/2}$. For $n = 10$, A. Garsia and E. Sergel found through computer search different sets of pairwise colliding permutations consisting of 251 elements (one less that the upper bound $\binom{10}{5} = 252$).

We found a code $E \subseteq S_5$ of 20 pairwise cyclic colliding permutations of 5 elements, which improves on the lower bound of 12 given by Proposition 2.1. This construction is structured as follows. Let

$$E' = \{x = (x_1, \dots, x_5) : x \text{ is a cyclic shift of } (1, 3, 2, *, *)\},$$

that is

$$E' = \{(1,3,2,*,*),(*,1,3,2,*),(*,*,1,3,2),(2,*,*,1,3),(3,2,*,*,1)\}.$$

E' consists of pairwise colliding permutations (as shown in Lemma 4.6 of [3]), hence cyclic colliding. One can "double" each partial permutation of E' filling the joker symbols $*$ of E' once with $4,5$, then with $5,4$ in the order: call the corresponding sets of 5 permutations $E'(4,5)$ and $E'(5,4)$ respectively: putting them together, one obtains 10 pairwise colliding permutations (hence cyclic colliding). In a similar way, we build

$$E'' = \{x = (x_1, \ldots, x_5) : x \text{ is a cyclic shift of } (5,3,4,*,*)\},$$

and "double" each of its elements filling the joker symbols $*$ of E' once with $1,2$, then with $2,1$ in this order, to obtain $E''(1,2)$, $E''(2,1)$, whose union leads to 10 colliding permutations. While the resulting set

$$F = E'(4,5) \cup E'(5,4) \cup E''(1,2) \cup E''(2,1)$$

is not a colliding code, it is surprisingly a cyclic colliding code.

We summarize the known values (or bounds) of the different considered types of T's up to 10 in the following table.

n	1	2	3	4	5	6	7	8	9	10
$T_2(n) = \binom{n}{\lfloor n/2 \rfloor}$	1	2	3	6	10	20	35	70	126	252
$T(n)$	1	2	3	6	10	20	35	70	126	$251 \leq ? \leq 252$
$T^*(n)$	1	2	6	6	$20 \leq ? \leq 30$	20	$40 \leq ? \leq 105$	70	$140 \leq ? \leq 378$	252

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