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On the ensemble of optimal dominating and locating-dominating codes in a graph

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ABSTRACT

Let G be a simple, undirected graph with vertex set V. For every $v \in V$, we denote by $N(v)$ the set of neighbours of v, and let $N[v] = N(v) \cup \{v\}$. A set $C \subseteq V$ is said to be a *dominating code* in G if the sets $N[v] \cap C$, $v \in V$, are all nonempty. A set $C \subseteq V$ is said to be a *locating-dominating code* in G if the sets $N[v] \cap C$, $v \in V \setminus C$, are all nonempty and distinct. The smallest size of a dominating (resp., locating-dominating) code in G is denoted by $d(G)$ (resp., $\ell(G)$).

We study the ensemble of all the different optimal dominating (resp., locating-dominating) codes C, i.e., such that $|C| = d(G)$ (resp., $|C| = \ell(G)$) in a graph G, and strongly link this problem to that of induced subgraphs of Johnson graphs.

1. Introduction

We introduce basic definitions and notation for graphs (for which we refer to, e.g., [2] and [4]), and for codes. Dominating codes constitute an old, large, classical topic (see, e.g., [5] or [6]); in the particular case when the graph is the hypercube, they are known as covering codes and have received a lot of attention in Coding Theory: see [3] and the on-line bibliography at [9], with 1000 references. Locating-dominating codes [12] are part of a larger class of codes which aim at distinguishing, in some ways, between vertices: watching systems, identifying, locating-dominating and discriminating codes, resolving sets, ...; they may have many applications and are a fast growing field, as show the 300 references in the on-line bibliography at [10], most of them published in the 21st century.

We denote by $G = (V, E)$ a simple, undirected graph with vertex set V and edge set E, where an *edge* between $x \in V$ and $y \in V$ is denoted by xy or yx. Two vertices linked by an edge are said to be *neighbours*. We denote by $N(v)$ the set of neighbours of the vertex v, and $N[v] = N(v) \cup \{v\}$. An *induced subgraph* of G is a graph with vertex set $X \subseteq V$ and edge set $\{uv \in E : u \in X, v \in X\}$. We say that two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic*, and write $G_1 \cong G_2$, if there is a bijection $\phi : V_1 \to V_2$ such that $xy \in E_1$ if, and only if, $\phi(x)\phi(y) \in E_2$ for all $x, y \in V$.

Whenever three vertices x, y, z are such that $x \in N[z]$ and $y \notin N[z]$, we say that z *separates* x and y in G (note that $z = x$ is possible). A set is said to separate x and y in G if it contains at least one vertex which does.

A *code* C is simply a subset of V, and its elements are called *codewords*. For each vertex $v \in V$, the *identifying set*

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of v, with respect to C, is denoted by $I_{G,C}(v)$ and is defined by

$$I_{G,C}(v) = N[v] \cap C.$$

We say that C is a *dominating* code in G if all the sets $I_{G,C}(v)$, $v \in V$, are nonempty.

We say that C is a *locating-dominating* code [12] if all the sets $I_{G,C}(v)$, $v \in V \setminus C$, are nonempty and distinct. In particular, any two non-codewords are separated by C. In the sequel, we shall use LD for locating-dominating.

We denote by $d(G)$ (respectively, $\ell(G)$) the smallest cardinality of a dominating (respectively, LD) code. Any dominating (respectively, LD) code C such that $|C| = d(G)$ (respectively, $|C| = \ell(G)$) is said to be *optimal*.

One application of LD codes is, for instance, fault diagnosis in multiprocessor systems: such a system can be modeled by a graph $G = (V, E)$ where V is the set of processors and E the set of links between processors. Assume that at most one processor is malfunctioning and we wish to test the system and locate the faulty processor. For this purpose, some processors (constituting the code) will be chosen and assigned the task of testing their neighbours. Whenever a selected processor (or codeword) detects a fault, it sends an alarm signal. We require that we can uniquely tell the location of the malfunctioning processor based on the information which ones of the codewords gave the alarm; under the assumption that the codewords work without failure, or that their only task is to test their neighbours (i.e., they are not considered as processors anymore) and that they perform this simple task without failure, then an LD code is what we need, because no two non-codewords have the same (nonempty) set of neighbours-codewords.

In this paper, we study the structure of the ensemble of all the optimal dominating codes and the ensemble of all the optimal LD codes of a graph. These ensembles are trivially collections of k-element subsets, or k-subsets, of V, for $k = d(G)$ or $k = \ell(G)$; we denote these ensembles by $\Xi(G)$ and $\Psi(G)$, respectively. Conversely, assume that $\mathcal{A}$ is a nonempty collection of some s different k-subsets $A_1, A_2, \ldots, A_s$ of $V_1 = \{1, 2, \ldots, n\}$. The question is: is there a graph G with vertex set V_1 such that $\mathcal{A}$ is equal to $\Xi(G)$ or $\Psi(G)$? When $3 \le k \le n-3$, the answer for *almost* all collections $\mathcal{A}$ is NO; indeed, there are $2^{\binom{n}{k}}$ such collections but only $2^{\binom{n}{2}}$ different graphs. However, we can ask the same question for a graph G with $n+m$ vertices, $m \ge 0$. And now the answer is YES: Theorem 2 below states that

> given any collection $\mathcal{A}$ of k-subsets of V_1, there is a positive integer m and a graph $G = (V, E)$ with $V = V_1 \cup V_2$, where $V_2 = \{n+1, \ldots, n+m\}$, such that $C \subseteq V$ is an optimal dominating code in G if, and only if, $C = A$ for some $A \in \mathcal{A}$.

So the ensemble of the optimal dominating codes of the graph G can be described by which k-set of vertices from V_1 we put in the code; now these k-sets are precisely the k-sets which belong to our target $\mathcal{A}$, and therefore the set $\Xi(G)$ is equivalent to $\mathcal{A}$. If, for any two k-subsets A_i and A_j in $\mathcal{A}$ we set

$$\delta(A_i, A_j) = |A_i \Delta A_j|,$$

where Δ stands for the symmetric difference, then, setting $C_i = A_i \cup S$ and $C_j = A_j \cup S$, we can see that $\delta(C_i, C_j) = \delta(A_i, A_j)$, i.e., G is such that $\Xi(G)$ has exactly the same symmetric difference distribution as the arbitrary collection $\mathcal{A}$ we started from.

Theorem 3 gives a similar result for LD codes, with similar consequences for $\Psi(G)$; also, the same kind of result is proved for identifying codes, which we do not define here, in [7].

Now, this establishes a sufficiently strong link, between the ensembles of the optimal dominating or LD codes of all graphs and the sets of k-subsets of n-sets, to connect our investigation to the following definition from [11] and the results related to it; see also [1].

Definition 1. Given positive integers k and n with $1 \le k \le n$, the Johnson graph $J(k, n)$ is the graph whose vertex set consists of all the k-subsets of $\{1, 2, \ldots, n\}$, with edges between two vertices sharing exactly $k-1$ elements.

A graph H is isomorphic to an induced subgraph of a Johnson graph if, and only if, it is possible to assign, for some k and n, a k-subset $S_v \subseteq \{1, 2, \ldots, n\}$ to each vertex v of H in such a way that distinct vertices have distinct corresponding k-sets, and vertices v and w are neighbours if, and only if, S_v and S_w share exactly $k-1$ elements. In this case, we say that H is an induced subgraph of a Johnson graph, or that H is a JIS for short.

We denote by $\mathcal{J}$ the set of all induced subgraphs of all Johnson graphs.

If we link two elements C_i and C_j in $\Xi(G)$ (respectively, $\Psi(G)$) if, and only if, $\delta(C_i, C_j) = 2$, then we obtain a graph which we denote by $\mathcal{N}(G)$ (respectively, $\mathcal{M}(G)$), and the set of all the graphs $\mathcal{N}(G)$ (respectively, $\mathcal{M}(G)$) is denoted by $\mathcal{N}$ (respectively, $\mathcal{M}$). Now, what Theorems 2 and 3 show as an immediate consequence is that

> every JIS belongs to $\mathcal{N}$, or: $\mathcal{J} = \mathcal{N}$;
> every JIS belongs to $\mathcal{M}$, or: $\mathcal{J} = \mathcal{M}$.

For examples of graphs which are JIS or not, we refer to [11], with a short overview in Section 3, but to our knowledge no classification is known.

2. Main results

Theorem 2. *Let $1 \le k \le n$ be an arbitrary integer, and assume that $\mathcal{A}$ is any nonempty collection of k-subsets of $V_1 = \{1, 2, \ldots, n\}$. Then there is a positive integer m and a graph G with vertex set $V = V_1 \cup V_2$, where $V_2 = \{n+1, n+2, \ldots, n+m\}$, such that $C \subseteq V$ is an optimal dominating code in G if, and only if, $C = A$ for some $A \in \mathcal{A}$.*

Proof. Denote by $\mathcal{B}$ the set of all $(k-1)$-subsets of V_1 together with all the k-subsets of V_1 that do not belong to $\mathcal{A}$; this set has size $\binom{n}{k-1} + \binom{n}{k} - |\mathcal{A}|$.

We begin the construction of G by taking n vertices $a_1, a_2, \ldots, a_n$, which we link together in all possible ways, so as to form the clique K_n (these vertices play the role of the vertices labelled by $1, 2, \dots, n$ in the statement of the theorem).

Let $B \in \mathcal{B}$ be arbitrary. Corresponding to B, we take $k+1$ new vertices $b_1(B), b_2(B), \dots, b_{k+1}(B)$, and link all of them to all the vertices a_i whose index i is not in B. When $k = 1$, we disregard the corresponding 0-element set.

The graph G thus constructed has $n + m$ vertices, with $m = (k+1)|\mathcal{B}|$.

First, it is easy to check that every code of the form $C(A) = \{a_i : i \in A\}$ is a dominating code in G when $A \in \mathcal{A}$. In particular, we have: $d(G) \le k$.

Conversely, assume that C is an optimal dominating code in G.

If for some $B \in \mathcal{B}$, each of the $k+1$ vertices $b_j(B)$ is a codeword, then $|C| \ge k+1$ and C is not optimal.

Therefore, for every B, there is at least one index j such that $b_j(B) \notin C$, so $b_j(B)$ must have a neighbour in C, and this neighbour is in $\{a_1, \dots, a_n\}$. This shows that $C \cap \{a_1, \dots, a_n\}$ cannot be the set $\{a_i : i \in B\}$, nor any of its subsets. Since this is true for every $B \in \mathcal{B}$, this proves that $|C| \ge k$ and so $d(G) = k$; moreover, if $|C| = k$, then C is of the form $C(A) = \{a_i : i \in A\}$ for some $A \in \mathcal{A}$.

This shows that the only optimal dominating codes in G are the codes $C(A) = \{a_i : i \in A\}$, $A \in \mathcal{A}$. □

Theorem 3. *Let $1 \le k \le n$ be an arbitrary integer, and assume that $\mathcal{A}$ is any nonempty collection of k-subsets of $V_1 = \{1, 2, \dots, n\}$. Then there is a positive integer m, a graph G with vertex set $V = V_1 \cup V_2$, where $V_2 = \{n+1, n+2, \dots, n+m\}$, and a set $S \subseteq V_2$ such that $C \subseteq V$ is an optimal locating-dominating code in G if, and only if, $C = A \cup S$ for some $A \in \mathcal{A}$.*

Proof. Denote by $\mathcal{B}$ the set of all $(k-1)$-subsets of V_1 together with all the k-subsets of V_1 that do not belong to $\mathcal{A}$; this set has size $\binom{n}{k-1} + \binom{n}{k} - |\mathcal{A}|$.

We begin the construction of G by taking n vertices $a_1, a_2, \dots, a_n$; these vertices play the role of the vertices labelled by $1, 2, \dots, n$ in the statement of the theorem.

Let $B \in \mathcal{B}$ be arbitrary. Corresponding to B, we form $k+1$ pairs $(b_j(B), c_j(B))$ of new vertices, $j = 1, 2, \dots, k+1$. Then, each a_i for which $i \in B$ is connected by an edge to the vertices $b_j(B)$ and $c_j(B)$ for all $j = 1, 2, \dots, k+1$; each a_i for which $i \notin B$ is connected by an edge to the vertices $b_j(B)$ for all $j = 1, 2, \dots, k+1$.

Now we choose a large enough integer K (in particular, $K \ge 6$), and take K new vertices $e_1, \dots, e_K$. Then we take $t = \binom{K}{4} + \binom{K}{3}$ new vertices $f_1, \dots, f_t$. No more vertices will be created, so the graph G will have $n + m$ vertices, with $m = 2(k+1)|\mathcal{B}| + K + t$.

For any vertex v, we say that its e-**signature** is the set of those e_i's that are within distance one from v. We now choose the edges between the e_i's and f_j's in such a way that every f_j has an e-signature of size three or four, and that all these signatures are different. Clearly, in the subgraph induced by the e_i's and f_j's (in the graph we have constructed so far), any LD code has size at least K, and the only LD code of size K consists of all the vertices e_i. Indeed, if even a single one of the e_i's, e_{i_0}, were missing, then we would have $\binom{K-1}{3} > K$ disjoint pairs of f_j's (each containing one vertex with an e-signature s of size three and one vertex with signature $s \cup \{e_{i_0}\}$ of size four) which could not be separated by the other $K-1$ vertices e_i, and so at least one element in each pair would have to belong to the LD code. On the other hand, the set $\{e_1, \dots, e_K\}$ is clearly an LD code.

Denote

$$V_{abc} = \bigcup_{B \in \mathcal{B}} \{b_j(B), c_j(B) : j = 1, 2, \dots, k+1\} \cup \{a_1, a_2, \dots, a_n\}$$

and

$$V_{ef} = \{e_1, e_2, \dots, e_K\} \cup \{f_1, f_2, \dots, f_t\}.$$

Finally, we add some edges between the vertices in V_{abc} and the vertices e_i. This is done in such a way that for each pair (j, B), where $j = 1, 2, \dots, k+1$ and $B \in \mathcal{B}$, the pair $(b_j(B), c_j(B))$ is assigned a unique e-signature, i.e., the two vertices $b_j(B)$, $c_j(B)$ both get the same signature, but the signatures are different for different pairs. Moreover, we assign each a_i, for $i = 1, 2, \dots, n$, an e-signature which is unique to that a_i. As K could be chosen to be arbitrarily large, this can be done in such a way that the e-signatures of all the vertices f_i also remain unique to them.

This completes the construction of our graph G.

Assume now that C is an optimal LD code in G. We have already seen that $|C \cap V_{ef}| \ge K$, and that equality implies that $C \cap V_{ef} = \{e_1, e_2, \dots, e_K\}$.

First, we suppose that for every $B \in \mathcal{B}$, there is at least one index j such that neither $b_j(B)$ nor $c_j(B)$ belong to C. Therefore $b_j(B)$ and $c_j(B)$ need to be separated by a codeword, and by the construction the only candidates are in $\{a_1, \dots, a_n\}$. This shows that $C \cap \{a_1, a_2, \dots, a_n\}$ cannot be the set $\{a_i : i \in B\}$, nor any of its subsets. As this is true for all $B \in \mathcal{B}$, we see that $|C \cap V_{abc}| \ge k$, and, moreover that if equality holds, then $C \cap V_{abc} = \{a_i : i \in A\}$ for some $A \in \mathcal{A}$.

Therefore $|C| \ge K + k$, and if equality holds, then C must be one of the codes

$$C(A) = \{e_1, e_2, \dots, e_K\} \cup \{a_i : i \in A\}$$

for some $A \in \mathcal{A}$.

The alternative is that for some B, each of the $k+1$ pairs $(b_j(B), c_j(B))$ contains at least one codeword; then $|C \cap V_{abc}| \ge k+1$ and C is not optimal.

Now it suffices to prove that every $C(A)$ is an LD code when $A \in \mathcal{A}$. This is clear: as the vertices a_i and f_j have unique signatures, it suffices to only consider the vertices in the pairs $(b_j(B), c_j(B))$. However, the signature assigned to that pair identifies the pair, and within a pair the vertices are separated by the vertices in $C \cap \{a_1, a_2, \dots, a_n\}$ as we saw above.

This shows that the only optimal LD codes in G are the codes $C(A) = \{e_1, e_2, \dots, e_K\} \cup \{a_i : i \in A\}$, $A \in \mathcal{A}$. □

The ensemble of the optimal LD codes of the graph G can be described by which k-set of vertices from V_1 we put in the code, since the other codewords (those in S) are common to all optimal LD codes; now these k-sets are precisely the k-sets which belong to our target $\mathcal{A}$, and therefore the set $\Psi(G)$ is equivalent to $\mathcal{A}$.

3. Some results on Johnson induced subgraphs

Some families of graphs are known to be JIS, some are known which are not JIS, but no characterization is available. Below, we summarize some of the results from [11]; for Cartesian products (see (d) below), we refer to [8].

Theorem 4.

(a) *[Prop. 4] All complete graphs and all cycles are JIS;*
(b) *All trees are JIS;*
(c) *[Prop. 6] A graph is a JIS if, and only if, all its connected components are JIS;*
(d) *[Prop. 7] The Cartesian product of two JIS is a JIS;*
(e) *[Prop. 12] Any graph obtained by removing one edge from the complete graph K_n, $n \geq 5$, is not a JIS;*
(f) *[Prop. 8] The complete bipartite graph $K_{2,3}$ is not a JIS.* □

The graph $K_{2,3}$ can be seen as two cycles of length four sharing three vertices; if we define the graph θ_n as the graph consisting of two cycles of length n sharing $n-1$ vertices, we have the following result from [11].

Theorem 5. *The graphs θ_4 and θ_5 are not JIS; all the graphs θ_n, $n \geq 6$, are JIS.* □

The q-ary n-dimensional hypercube is another graph which is JIS, for all $q \geq 2$ and $n \geq 1$; indeed, the q-ary words of length n in Z_q^n can be transformed into binary sequences of length qn, containing exactly n ones, applying the mapping $\phi : Z_q \to Z_2^q$, with $\phi(0) = e_1$ and $\phi(i) = e_{i+1}$ for $i \in \{1, 2, \ldots, q-1\}$, where e_i has exactly one "1" in position i, so that Z_q^n can be seen as a collection of n-subsets of $\{1, 2, \ldots, qn\}$.

References

[1] B. Alspach, Johnson graphs are Hamiltonian-connected, Ars Math. Contemp. 6 (2013) 21–23.
[2] C. Berge, Graphes, Gauthier-Villars, Paris, 1983, English translation: North-Holland Publishing Co., Amsterdam, 1985.
[3] G.D. Cohen, I.S. Honkala, S. Litsyn, A.C. Lobstein, Covering Codes, Elsevier, Amsterdam, 1997.
[4] R. Diestel, Graph Theory, Springer-Verlag, Berlin, 2005.
[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[6] M.A. Henning, A. Yao, Total Domination in Graphs, Springer-Verlag, New York, 2013.
[7] I. Honkala, O. Hudry, A. Lobstein, On the ensemble of optimal identifying codes in a twin-free graph, Cryptogr. Commun., to appear.
[8] W. Imrich, S. Klavžar, D.F. Rall, Topics in Graph Theory: Graphs and Their Cartesian Product, A.K. Peters, Wellesley, MA, 2008.
[9] A. Lobstein, A bibliography on covering radius, http://perso.telecom-paristech.fr/~lobstein/bib-a-jour.pdf.
[10] A. Lobstein, A bibliography on watching systems, identifying, locating-dominating and discriminating codes in graphs, http://perso.telecom-paristech.fr/~lobstein/debutBIBidetlocdom.pdf.
[11] R. Naimi, J. Shaw, Induced subgraphs of Johnson graphs, Involve 5 (2012) 25–37.
[12] D.F. Rall, P.J. Slater, On location-domination numbers for certain classes of graphs, Congr. Numer. 45 (1984) 97–106.