Contrast re-enhancement of Total-Variation regularization jointly with the Douglas-Rachford iterations

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***Abstract*—Restoration of a piece-wise constant signal can be performed using anisotropic Total-Variation (TV) regularization. Anisotropic TV may capture well discontinuities but suffers from a systematic loss of contrast. This contrast can be re-enhanced in a post-processing step known as least-square refitting. We propose here to jointly estimate the refitting during the Douglas-Rachford iterations used to produce the original TV result. Numerical simulations show that our technique is more robust than the naive post-processing one.**

I. Introduction

We consider the reconstruction of a 2D signal identified as a vector $u_0 \in \mathbb{R}^N$ from its noisy observation $f = \Phi u_0 + w \in \mathbb{R}^P$ with $w \in \mathbb{R}^P$ a zero-mean noise component and $\Phi \in \mathbb{R}^{P \times N}$ a linear operator accounting for a loss of information (e.g., low-pass filter). Anisotropic TV regularization writes, for $\lambda > 0$, as [1]

$$u^{\mathrm{TV}} \in \underset{u \in \mathbb{R}^N}{\operatorname{argmin}} \ \tfrac{1}{2}\|\Phi u - f\|^2 + \lambda \|\nabla u\|_1, \tag{1}$$

with $\nabla u \in \mathbb{R}^{2N}$ being the concatenation of vertical and horizontal components of the discrete gradient vector field of u, and $\|\nabla u\|_1 = \sum_i |(\nabla u)_i|$ being a sparsity promoting term. Anisotropic TV is known to recover piece-wise constant signals. However, even though the discontinuities can be correctly recovered in some cases, the amplitudes of u^{TV} are known to suffer from a loss of contrast compared to u_0 [2].

II. Least-Square refitting problem

A simple technique to correct this effect, known as least-square refitting, consists in enhancing the amplitudes of u^{TV} while leaving unchanged the set of discontinuities, as

$$\tilde{u}^{\mathrm{TV}} \in \underset{u\ ;\ \mathrm{supp}(\nabla u) \subset \mathrm{supp}(\nabla u^{\mathrm{TV}})}{\operatorname{argmin}} \|\Phi u - f\|^2 \tag{2}$$

where, for $x \in \mathbb{R}^{2N}$, $\mathrm{supp}(x) = \{i \in [2N]\ ;\ \|x_i\| \neq 0\}$ denotes the support of x. Post-refitting identifies $\mathrm{supp}(\nabla u^{\mathrm{TV}})$ and solves (2) [3], typically with a conjugate gradient. However, u^{TV} is usually obtained thanks to a converging sequence u^k, and unfortunately, $\mathrm{supp}(\nabla u^k)$ can be far from $\mathrm{supp}(\nabla u^{\mathrm{TV}})$ even though u^k can be made arbitrarily close to u^{TV}. Such erroneous support identifications can lead to results that strongly deviates from the solution $\tilde{u}^{\mathrm{TV}}$.

III. Joint refitting with Douglas-Rachford

To alleviate this difficulty, we build a sequence $\tilde{u}^k$ jointly with u^k that converges towards a solution $\tilde{u}^{\mathrm{TV}}$. We consider the Douglas-Rachford sequence u^k applied to the splitting TV reformulation [4] given by

$$u^{\mathrm{TV}} \in \underset{u \in \mathbb{R}^N}{\operatorname{argmin}} \ \min_{z \in \mathbb{R}^{N \times 2}} \ \tfrac{1}{2}\|\Phi u - f\|^2 + \lambda \|z\|_{1,2} + \iota_{\{z,u\ ;\ z = \nabla u\}}(z, u)$$

where ι_S is the indicator function of a set S. This leads to the proposed algorithm given, for $\tau > 0$ and $\beta \geqslant 0$, by Eq. (3) (see right column). The sequence u^k is exactly the Douglas-Rachford sequence converging towards a solution u^{TV} [5]. Regarding $\tilde{u}^k$, we prove the following.

Theorem 1. *Let $\alpha > 0$ be the minimum non zero value of $|(\nabla u)_i|$, $i \in [2N]$. For $0 < \beta < \alpha\lambda$, $\tilde{u}^k$ converges towards a solution $\tilde{u}^{\mathrm{TV}}$.*

Sketch of proof: As u^k converges towards a solution u^{TV}, for k large enough, we get after few manipulations and triangle inequalities that

$$\left\{ \begin{array}{l} \mu^{k+1} = (\mathrm{Id} + \Delta)^{-1}(2u^k - \mu^k - \mathrm{div}(2z^k - \zeta^k))/2 + \mu^k/2, \\ \tilde{\mu}^{k+1} = (\mathrm{Id} + \Delta)^{-1}(2\tilde{u}^k - \tilde{\mu}^k - \mathrm{div}(2\tilde{z}^k - \tilde{\zeta}^k))/2 + \tilde{\mu}^k/2, \\ \zeta^{k+1} = \nabla \mu^{k+1}, \qquad \tilde{\zeta}^{k+1} = \nabla \tilde{\mu}^{k+1}, \\ u^{k+1} = \mu^{k+1} + \tau \Phi^t (\mathrm{Id} + \tau \Phi \Phi^t)^{-1}(f - \Phi \mu^{k+1}), \\ \tilde{u}^{k+1} = \tilde{\mu}^{k+1} + \tau \Phi^t (\mathrm{Id} + \tau \Phi \Phi^t)^{-1}(f - \Phi \tilde{\mu}^{k+1}), \\ z^{k+1} = \Psi_{\zeta^{k+1}}(\zeta^{k+1}, \lambda), \quad \tilde{z}^{k+1} = \Pi_{\zeta^{k+1}}(\tilde{\zeta}^{k+1}, \lambda) \end{array} \right. \tag{3}$$

$$\text{where} \quad \Psi_\zeta(\zeta, \lambda)_i = \left\{ \begin{array}{ll} 0 & \text{if} \quad |\zeta_i| \leqslant \tau\lambda, \\ \zeta_i - \tau\lambda \,\mathrm{sign}\, \zeta_i & \text{otherwise} \end{array} \right.$$

$$\text{and} \quad \Pi_\zeta(\tilde{\zeta}, \lambda)_i = \left\{ \begin{array}{ll} 0 & \text{if} \quad |\zeta_i| \leqslant \tau\lambda + \beta, \\ \tilde{\zeta}_i & \text{otherwise.} \end{array} \right.$$

Fig. 1. Damaged image, result of TV, post-refitting and our joint-refitting.

$\{i\ ;\ |\zeta_i^k| > \tau\lambda + \beta\} = \mathrm{supp}(\nabla u^{\mathrm{TV}})$ (for the given range of β). As a result, for k large enough, the sequence (3) can be rewritten by substituting $\Pi_\zeta(\cdot, \lambda)$ by the projector onto $\{u\ ;\ \mathrm{supp}(u) \subset \mathrm{supp}(\nabla u^{\mathrm{TV}})\}$ which is exactly the Douglas-Rachford sequence for the refitting problem (2) which is provably converging towards a solution $\tilde{u}^{\mathrm{TV}}$ [5]. □

IV. Results and discussion

Figure 1 shows results on an 8bits image damaged by a Gaussian blur of 2px and white noise $\sigma = 20$. The parameter β is chosen as the smallest positive value up to machine precision. While TV reduces the contrast, refitting recovers the original amplitudes and keep unchanged the discontinuities. Post-refitting offers comparable results to ours except for suspicious oscillations due to wrong support identification.

Being computing during the Douglas-Rachford iterations, our refitting strategy is free of post-processing steps such as support identification. It is moreover easy to implement and can be used likewise for other ℓ_1 analysis penalties. Extensions of this approach for isotropic TV or block sparsity regularizations are under investigation.

References

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