Asymptotics of superposition of point processes

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Abstract. The characteristic independence property of Poisson point processes gives an intuitive way to explain why a sequence of point processes becoming less and less repulsive can converge to a Poisson point process. The aim of this paper is to show this convergence for sequences built by superposing repulsive point processes. We use Papangelou intensities and Stein's method to prove this result with a topology based on total variation distance.

Keywords: Stochastic Geometry, Ginibre point process, β-Ginibre point process, Poisson point process, Stein's method

1 Motivations

The primary motivation of this work was the following. Consider the locations of base stations (BS), i.e. antennas, of the mobile network in Paris. If we have a look at the global process of all base stations of all operators and for all operating frequencies, we obtain the left picture of Figure 1. It turns to be compatible with the null hypothesis of being a Poisson process. However, if we look at the positions of base stations deployed by one operator, in one frequency band, we get a picture similar to the right picture of Figure 1. It was shown in [5] that this deployment is statistically compatible with a point process with repulsion, called β-Ginibre process.

Fig. 1. On the left, positions of all BS in Paris. On the right, locations of BS for one frequency band.

When superposing a large number of independent processes with internal repulsion but few points, it is intuitively clear that the resulting process does not exhibit strong interdependencies between its atoms and should thus resemble a Poisson process. This is this intuition we wanted to quantify by determining how fast does the convergence hold. It is often clear by looking at the Laplace transforms that a superposition of processes converge to a Poisson process, however, this does not yield a convergence rate. We here use the Stein-Dirichlet-Malliavin method, developed in [1, 3], to precise this rate. It turns out that the pertinent characteristics of the point processes to be considered is their Papangelou intensity, see [4] and references therein. We show here that the Kantorovitch-Rubinstein between a Poisson point process and any other point process is controlled by the L^1 distance of their Papangelou intensity, thus generalizing the property that the distance between two Poisson process is controlled by the L^1 distance between their control measure [2]. This result is then applied to several situations involving superpositions and dilations of point processes. This paper is organized as follows: In Section 2, we recall the basics of point processes theory and introduce our model of choice, the repulsive point processes. Section 3 is devoted to the explanation of the Stein-Dirichlet-Malliavin method and how we get the main theorem. In Section 4, we apply this result to superposition of repulsive point processes. The proofs are given in Appendix.

2 Preliminaries

2.1 Point processes

Let $\mathbb{Y}$ be a normed vector space and $\mathcal{F}_{\mathbb{Y}}$ its Borel algebra, $N_{\mathbb{Y}}$ the space of all locally finite subsets (configurations) in $\mathbb{Y}$, $\widehat{N}_{\mathbb{Y}}$ the space of finite subsets in $\mathbb{Y}$, and μ a diffuse and locally finite measure on $\mathbb{Y}$. $\mathbb{N}$ henceforth denoted the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

A *point process* in $\mathbb{Y}$ is a random integer-valued positive and locally finite measure Φ on $\mathbb{Y}$ [6]. If Φ almost-surely assigns at most measure 1 to singletons, it is a *simple* point process, and in this case, for any measurable set A, $\Phi(A)$ represents the number of points of the process that fall in A. Note that a point process Φ can be seen as a random configuration or reduced to his probability distribution $\mathbb{P}_{\Phi}$. Moreover, the *intensity* (or *measure control*) of a point process Φ is defined as the measure $A \in \mathcal{F}_{\mathbb{Y}} \mapsto \mathbf{E}[\Phi(A)]$ on $\mathbb{Y}$.

The *μ-sample measure* S_μ is defined for any measurable $f : \widehat{N}_{\mathbb{Y}} \to \mathbb{R}_+$ by:

$$\int_{\widehat{N}_{\mathbb{Y}}} f(\alpha) S_\mu(\mathrm{d}\alpha) = \sum_{k=0}^{+\infty} \frac{1}{k!} \int_{\mathbb{Y}^k} f(\{x_1, \ldots, x_k\}) \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_k). \tag{1}$$

The *correlation function* $\rho : \widehat{N}_{\mathbb{Y}} \to \mathbb{R}_+$ of a point process Φ with respect to the probability measure $\mathbb{P}_{\Phi}$ is defined for any measurable bounded function $f : \widehat{N}_{\mathbb{Y}} \to \mathbb{R}_+$ by:

$$\int_{N_{\mathbb{Y}}} \sum_{\substack{\alpha \in \widehat{N}_{\mathbb{Y}} \\ \alpha \subset \xi}} f(\alpha) \mathbb{P}_{\Phi}(\mathrm{d}\xi) = \int_{\widehat{N}_{\mathbb{Y}}} f(\alpha) \rho(\alpha) S_\mu(\mathrm{d}\alpha) \tag{2}$$

and characterizes the law of Φ ; in particular, the intensity measure is given by $A \in \mathcal{F}_{\mathbb{Y}} \mapsto \int_A \rho(\{x\})\mu(\mathrm{d}x)$, and $x \in \mathbb{Y} \mapsto \rho(\{x\})$ designs the *density* of Φ with respect to μ. Hence, the correlation function of a Poisson point process (PPP) with control measure $M(\mathrm{d}x) = m(x)\mu(\mathrm{d}x)$ on $\mathbb{Y}$ is given by $\rho(\xi) = \prod_{x\in\xi} m(x)$.

We say that $c : \mathbb{Y} \times N_{\mathbb{Y}} \to \mathbb{R}_+$ is the *Papangelou intensity* of a point process Φ w.r.t. $\mathbb{P}_\Phi$ on $\mathbb{Y}$ if, for any measurable function $f : \mathbb{Y} \times N_{\mathbb{Y}} \to \mathbb{R}_+$,

$$\int_{N_{\mathbb{Y}}} \sum_{x\in\xi} f(x, \xi \setminus \{x\})\mathbb{P}_\Phi(\mathrm{d}\xi) = \int_{\mathbb{Y}} \int_{N_{\mathbb{Y}}} c(x,\xi) f(x,\xi)\mathbb{P}_\Phi(\mathrm{d}\xi)\mu(\mathrm{d}x). \quad (3)$$

It immediatly comes from (2) and (3) that for any $x \in \mathbb{Y}$,

$$\rho(x) = \mathbf{E}[c(x, \Phi)], \quad (4)$$

and if Φ is a *finite* point process (i.e. such that $\Phi(\mathbb{Y}) < +\infty$ a.s.), then by (3),

$$\mathbb{P}(|\Phi| = 1) = \int_{\mathbb{Y}} c(x, \varnothing)\mu(\mathrm{d}x)\mathbb{P}(|\Phi| = 0). \quad (5)$$

We henceforth say that a point process Φ is *repulsive* if its Papangelou intensity c verifies, for any $\xi \in N_{\mathbb{Y}}$ and any $x \in \mathbb{Y}$, the inequality $c(x, \xi) \le c(x, \varnothing)$. By theorem 3.1. in [4], a determinantal point process reduced to a compact set is repulsive, and so is a Gibbs point process.

2.2 Determinantal point processes

For the applications we have in mind, we introduce now the notion of determinantal point processes, for details we refer to **?**. A process of this kind is characterized by a reference measure μ on $\mathbb{Y}$ and an Hilbert-Schmidt linear map K from $L^2(Y, \mu; \mathbb{C})$ into itself satisfying the following properties:

- K is positive Hermitian.
- The discrete spectrum of K is included in $[0, 1)$.
- K is a locally trace-class: For any compact $\Lambda \subset \mathbb{Y}$, $K_\Lambda = P_\Lambda K P_\Lambda$ (where P_Λ is the orthogonal projection of $L^2(\mathbb{Y}, \mu; \mathbb{C})$ to $L^2(\Lambda, \mu; \mathbb{C})$), the restriction of K to $L^2(\Lambda, \mu; \mathbb{C})$, is trace class.

Since K is Hilbert-Schmidt, there exists a kernel, which we still denote by K, from $\mathbb{Y} \times \mathbb{Y}$ into $\mathbb{C}$, such that for any $x \in \mathbb{Y}$,

$$Kf(x) = \int_{\mathbb{Y}} K(x, y)f(y)\mu(dy).$$

Together with K, there is another operator of importance, usually denoted by J and defined as $J = (I - K)^{-1}K$. Since K is hermitian, there exists a complete orthonormal basis $(h_j,\, j \in \mathbb{N})$ of $L^2(\mathbb{Y}, \mu; \mathbb{C})$ and a sequence $(\lambda_j,\, j \in \mathbb{N}) \subset [0, 1)^{\mathbb{N}}$ such that for all $f \in L^2(\mathbb{Y}, \mu; \mathbb{C})$,

$$Kf = \sum_{j=1}^{+\infty} \lambda_j \langle f, h_j\rangle_{L^2(\mu)} h_j, \;\; Jf = \sum_{j=1}^{+\infty} \frac{\lambda_j}{1-\lambda_j} \langle f, h_j\rangle_{L^2(\mu)} h_j,$$

and then, for all $x, y \in \mathbb{Y}$,

$$K(x,y) = \sum_{j=1}^{+\infty} \lambda_j h_j(x) h_j(y),\ J(x,y) = \sum_{j=1}^{+\infty} \frac{\lambda_j}{1-\lambda_j} h_j(x) h_j(y).$$

The determinantal point process $\mathrm{DPP}(K,\mu)$ is then defined by its correlation functions (see **?**):

$$\rho(\{x_1, \cdots, x_k\}) = \det(K(x_i,x_j),\ 1 \leq i,j \leq k).$$

From 4, we know that

$$c(x_0, \{x_1, \cdots, x_k\}) = \frac{\det(J(x_i,x_j),\ 0 \leq i,j \leq k)}{\det(J(x_i,x_j),\ 1 \leq i,j \leq k)}.$$

2.3 Thinned and rescaled point processes

Let Φ be a point process on $\mathbb{Y}$. If $\varepsilon \in [0;1]$, we associate to Φ the *ε-thinned point process* $t_\varepsilon(\Phi)$ obtained by retaining, independently and with probability ε, each point of Φ.

If $\mathbb{Y} = \mathbb{R}^d$ and γ is a positive real number, we associate to Φ the *γ-rescaled point process* $r_\gamma(\Phi)$ obtained by applying a dilation of magnitude $\gamma^{1/d}$ to each point of Φ. Note that this modifies the intensity measure of Φ by a factor γ.

For $\beta \in (0;1]$, we associate to Φ the β-point process $r_{\beta^{-1}}(t_\beta(\Phi))$ obtained by combining a β-thinning and a β^{-1}-rescaling, in order to conserve the intensity measure of Φ. Their respective correlation functions are provided by the following proposition.

Proposition 1. *Let Φ be a point process on $\mathbb{Y}$ with correlation function ρ, and $\varepsilon \in [0;1]$. Then, the function correlation of $t_\varepsilon(\Phi)$ is given for any $\alpha \in \widehat{N}_{\mathbb{Y}}$ by*

$$\rho_{t_\gamma(\Phi)}(\alpha) = \varepsilon^{|\alpha|}\rho(\alpha).$$

Moreover, if $\mathbb{Y} = \mathbb{R}^d$ and $\gamma > 0$, the correlation function σ_γ of $r_\gamma(\Phi)$ is given for any $\alpha \in \widehat{N}_{\mathbb{Y}}$ by

$$\rho_{r_\gamma(\Phi)}(\alpha) = \gamma^{|\alpha|}\rho(\gamma^{\frac{1}{d}}\alpha).$$

3 Kantorovitch-Rubinstein distance and Stein's method

The total variation distance between two measures ν_1 and ν_2 on $\mathbb{Y}$ is defined by

$$d_{\mathrm{TV}}(\nu_1,\nu_2) := \sup_{\substack{A \in \mathcal{F}_{\mathbb{Y}} \\ \nu_1(A),\nu_2(A) < \infty}} |\nu_1(A) - \nu_2(A)|.$$

We say that a measurable map $F : N_{\mathbb{Y}} \to \mathbb{R}$ is 1-Lipschitz if

$$|F(\phi_1) - F(\phi_2)| \leq \mathrm{d}_{TV}(\phi_1,\phi_2) \text{ for all } \phi_1, \phi_2 \in N_{\mathbb{Y}}.$$

We denote by Lip_1 the set of bounded 1-Lipschitz maps. The Kantorovich-Rubinstein distance between two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$ on $N_{\mathbb{Y}}$ is defined by

$$d_{\mathrm{KR}}(\mathbb{P}_1,\mathbb{P}_2)=\sup_{F\in\mathrm{Lip}_1}\Big|\int_{N_{\mathbb{Y}}}F(\phi)\,\mathbb{P}_1(\mathrm{d}\phi)-\int_{N_{\mathbb{Y}}}F(\phi)\,\mathbb{P}_2(\mathrm{d}\phi)\Big|. \tag{6}$$

According to [3Proposition 2.1], the topology induced by this distance coincides with the topology of narrow convergence of probability measures on $N_{\mathbb{Y}}$. Our goal is to evaluate the distance between some probability measure on $N_{\mathbb{Y}}$ and $\mathbb{P}_M$, the distribution of a Poisson point process of control measure M on $\mathbb{Y}$. We assume henceforth that M has a finite mass, i.e. $M(\mathbb{Y})<\infty$. We use the Stein-Dirichlet-Malliavin method which we describe roughly now, for details we refer to [3].

The Glauber process $(G_t,\,t\ge 0)$ associated to $\mathbb{P}_M$ is the $\widehat{N}_{\mathbb{Y}}$-valued Markov process whose generator is given by

$$LF(\phi):=\int_{\mathbb{Y}}(F(\phi+\delta_y)-F(\phi))\,M(\mathrm{d}y)+\sum_{y\in\phi}(F(\phi-\delta_y)-F(\phi)),\ \phi\in\widehat{N}_{\mathbb{Y}},$$

where $F:N_{\mathbb{Y}}\to\mathbb{R}$ is a measurable and bounded function. Since M is a finite measure, the dynamics of G is described as follows: Let $G(0)=\phi$ and consider a Poisson process on the half-axis of intensity $M(\mathbb{Y})$. We denote by $(T_n,\,n\ge 1)$ the arrival times of this process. At each T_n, $G(T_n)=G(T_n^-)+\delta_{Y_n}$ where Y_n is chosen according to M, independently of everything else. All the particles, be they present at the origin or born after, have a lifetime which follows an exponential distribution of parameter 1, independent of everything else. Then, $G(t)$ is the point process of living particles at time t. We denote by $(P_t,\,t\ge 0)$ its semi-group:

$$P_tF(\phi)=\mathbf{E}\left[F(G(t))\,|\,G(0)=\phi\right].$$

This Markov process, or at least its semi-group, has two attractive features:

- It is ergodic: $\lim_{t\to\infty}P_tF(\phi)=\int_{N_{\mathbb{Y}}}F(\zeta)\mathbb{P}_M(\mathrm{d}\zeta)$ for all $\phi\in N_{\mathbb{Y}}$.
- If we define the operator D by

  $$D_yF(\phi)=F(\phi+\delta_y)-F(\phi),$$

  for any $y\in\mathbb{Y}$ and $\phi\in N_{\mathbb{Y}}$, we have

  $$D_yP_tF(\phi)=e^{-t}P_tD_yF(\phi),$$

  for all $t\ge 0$, $y\in\mathbb{Y}$ and $\phi\in N_{\mathbb{Y}}$.

As a consequence of the ergodicity and of the markovianity of P, we have the Stein-Dirichlet representation formula, see [3]: For any probability measure $\mathbb{P}$ on $N_{\mathbb{Y}}$,

$$\int_{N_{\mathbb{Y}}}F(\phi)\,\mathbb{P}(\mathrm{d}\phi)-\int_{N_{\mathbb{Y}}}F(\phi)\,\mathbb{P}_M(\mathrm{d}\phi)=\int_{N_{\mathbb{Y}}}\int_0^\infty LP_sh(\phi)\,ds\ \mathbb{P}(\mathrm{d}\phi).$$

Theorem 1. *Let $\mathbb{P}$ be a finite point process on $\mathbb{Y}$ with Papangelou intensity c, and $\mathbb{P}_M$ the distribution of a Poisson point process with finite control measure $M(dy) = m(y)\mu(dy)$ on $\mathbb{Y}$. Then, we have the following upper bound:*

$$d_{KR}(\mathbb{P},\, \mathbb{P}_M) \le \int_\Lambda \int_{N_\mathbb{Y}} |m(y) - c(y,\phi)|\nu(d\phi)\mu(dy).$$

Proof. Starting from the expression of L, we have

$$\begin{aligned}
&\int_{N_\mathbb{Y}} F(\phi)\,\mathbb{P}(\mathrm{d}\phi) - \int_{N_\mathbb{Y}} F(\phi)\,\mathbb{P}_M(\mathrm{d}\phi)\\
&= \int_{N_\mathbb{Y}} \int_0^\infty \int_\mathbb{Y} D_y P_s F(\phi)\, m(y)\mu(\mathrm{d}y)\, ds\ \mathbb{P}(\mathrm{d}\phi)\\
&\qquad\qquad + \int_{N_\mathbb{Y}} \int_0^\infty \sum_{y\in\phi} P_s F(\phi - \delta_y) - P_s F(\phi)\, ds\ \mathbb{P}(\mathrm{d}\phi).
\end{aligned}$$

By the very definition of the Papangelou intensity,

$$\begin{aligned}
&\int_{N_\mathbb{Y}} \int_0^\infty \sum_{y\in\phi} P_s F(\phi - \delta_y) - P_s F(\phi)\, ds\ \mathbb{P}(\mathrm{d}\phi)\\
&\qquad\qquad = - \int_{N_\mathbb{Y}} \int_0^\infty \int_Y D_y P_s F(\phi) c(y,\, \phi)\mu(\mathrm{d}y)\, ds\ \mathbb{P}(\mathrm{d}\phi).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{N_\mathbb{Y}} F(\phi)\,\mathbb{P}(\mathrm{d}\phi) - \int_{N_\mathbb{Y}} F(\phi)\,\mathbb{P}_M(\mathrm{d}\phi)\\
&\qquad = \int_{N_\mathbb{Y}} \int_0^\infty \int_Y D_y P_s F(\phi)(m(y) - c(y,\, \phi))\mu(\mathrm{d}y)\, ds\ \mathbb{P}(\mathrm{d}\phi).
\end{aligned}$$

In view of the commutation relationship between D_y and P_s, since F Lipschitz entails that $|D_y F(\phi)| \le 1$ for any (y, ϕ), we get

$$|D_y P_s F(\phi)| = e^{-s} |P_s D_y F(\phi)| \le e^{-s} P_s \mathbb{1} = e^{-s}.$$

The result then follows.

4 Application to superpositions of repulsive point processes

For all $n \in \mathbb{N}$, we consider $\Phi_{n,1}, \ldots, \Phi_{n,n}$, n independent, finite and repulsive point processes on $\mathbb{Y}$ and Φ_n their superposition. For all $i \in \{1, \ldots, n\}$, $\rho_{n,i}$ denotes the correlation function of $\Phi_{n,i}$, and for any $k \in \mathbb{N}_0$, we use the notation $p_{n,i,k} := \mathbb{P}(|\Phi_{n,i}| = k)$. We furthermore suppose that the two following assumptions are verified:

(A1) There exists a measurable function m and a positive constant $K_1 > 0$ such that for any $n \in \mathbb{N}$,

$$R_n := \int_{\mathbb{Y}} |\sum_{i=1}^{n} \rho_{n,i}(x) - m(x)|\mu(\mathrm{d}x) \leq \frac{K_1}{n}.$$

(A2) There exists a positive constant $K_2 > 0$ such that for any $n \in \mathbb{N}$ and any $i \in \{1, \ldots, n\}$,

$$\int_{\mathbb{Y}} \rho_{n,i}(x)\mu(\mathrm{d}x) \leq \frac{K_2}{n} \int_{\mathbb{Y}} \rho(x)\mu(\mathrm{d}x).$$

Let us give a few comments on these assumptions: (A1) ensures that the intensity of the superposition converges to the intensity $M(\mathrm{d}x) = m(x)\mathrm{d}x$ and (A2) allows to control the intensities of the elements of the superposition.

The Papangelou intensity of the superposition is expressed with help from Papangelou intensities $c_{n,i}$ of the $\Phi_{n,i}$ in the following lemma:

Lemma 1. *For all $n \in \mathbb{N}$, Φ_n admits a Papangelou intensity c_n given for all $y \in \mathbb{Y}$, $\xi \in N_{\mathbb{Y}}$ by:*

$$c_n(y, \Phi_n) = \sum_{i=1}^{n} c_{n,i}(y, \Phi_{n,i}).$$

The following proposition states the convergence of (Φ_n) with convergence speed. Its proof uses upper bound given by Stein's method.

Proposition 2. *Let be a sequence (Φ_n) built for any $n \in \mathbb{N}$ by superposition of n independent, finite and repulsive point processes $\Phi_{n,1}, \ldots, \Phi_{n,n}$. If (A1) and (A2) holds, then (Φ_n) converges with respect to Kantorovitch-Rubinstein distance to a Poisson point process π_M with control measure $M(dx) = m(x)\mu(dx)$. In addition, there exists a positive constant K such that, for any $n \in \mathbb{N}$,*

$$d_{KR}(\Phi_n, \pi_M) \leq \frac{K}{n}.$$

5 Application to β-determinantal processes

Let K be the kernel of a DPP on $\mathbb{R}^d$, and Λ a compact subset of $\mathbb{R}^d$. Suppose that $(\beta_n)_{n\in\mathbb{N}}$ converges to 0 and that, for all $n \in \mathbb{N}$, Φ_n is the DPP with kernel K_n defined by

$$K_n : (x, y) \in \mathbb{Y} \times \mathbb{Y} \mapsto K\Big(\frac{x}{\sqrt{\beta_n}}, \frac{x}{\sqrt{\beta_n}}\Big)\mathbf{1}_{\Lambda\times\Lambda}(x, y). \tag{7}$$

Let be the sequence $(R_n, n \in \mathbb{N})$ defined for all $n \in \mathbb{N}$ by

$$R_n = \int_E |K_n(y, y) - K(0, 0)|\mu(\mathrm{d}y).$$

Proposition 3. *Suppose*

$$\lim_{n \to +\infty} R_n = 0,$$

then the sequence (Φ_n) *converges strongly to an homogeneous PPP with intensity* $M(dy) = K(0,0)\mu(dy)$ *and, more precisely, there exists a constant* $C > 0$ *such that, for all* $n \in \mathbb{N}$,

$$d_{KR}(\Phi_n, \pi_M) \leq C(\beta_n + R_n).$$

6 Conclusion

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A Proofs of Proposition 1, Theorem 1, Lemmas 2 and 1

A.1 Proof of Proposition 1

For any measurable bounded function $f : \widehat{N}_{\mathbb{Y}} \to \mathbb{R}_+$,

$$\begin{aligned}\mathbf{E}[\sum_{\substack{\alpha\subset t_\varepsilon(\Phi)\\ \alpha\in\widehat{N}_{\mathbb{Y}}}} f(\alpha)] &= \sum_{n=0}^{+\infty}\mathbf{E}[\sum_{\substack{\alpha\subset t_\varepsilon(\Phi)\\ |\alpha|=n}} f(\alpha)]\\ &= \sum_{n=0}^{+\infty}\mathbf{E}[\sum_{\substack{\alpha\subset \Phi\\ |\alpha|=n}} \varepsilon^n f(\alpha)],\end{aligned}$$

where the last equality is obtained using the fact that ε^n is the probability to have n successes among n Bernoulli trials with success probability ε. Hence, using equation (2) defining correlation function,

$$\begin{aligned}\mathbf{E}[\sum_{\substack{\alpha\subset t_\varepsilon(\Phi)\\ \alpha\in\widehat{N}_{\mathbb{Y}}}} f(\alpha)] &= \mathbf{E}[\sum_{\alpha\subset\Phi} \varepsilon^{|\alpha|} f(\alpha)]\\ &= \int_{\widehat{N}_{\mathbb{Y}}} f(\alpha)\varepsilon^{\alpha}\rho(\alpha)S_\mu(\mathrm{d}\alpha),\end{aligned}$$

which ends to show the first assertion.

Let us consider the γ-rescaled point process. Using equations (1) and (2), for any measurable bounded function $f : \widehat{N}_{\mathbb{Y}} \to \mathbb{R}_+$,

$$\begin{aligned}\mathbf{E}[\sum_{\substack{\alpha\subset r_\gamma(\Phi)\\ \alpha\in\widehat{N}_{\mathbb{Y}}}} f(\alpha)] &= \mathbf{E}[\sum_{\substack{\omega\subset\Phi\\ \omega\in\widehat{N}_{\mathbb{Y}}}} f(\gamma^{-\frac{1}{d}}\omega)]\\ &= \int_{\widehat{N}_{\mathbb{Y}}} f(\gamma^{-\frac{1}{d}}\alpha)\rho(\alpha)S_\mu(\mathrm{d}\alpha)\\ &= \sum_{k=0}^{+\infty}\frac{1}{k!}\int_{\mathbb{Y}^k} f(\{\gamma^{-\frac{1}{d}}x_1,\dots,\gamma^{-\frac{1}{d}}x_k\})\rho(\{x_1,\dots,x_k\})\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_k)\\ &= \sum_{k=0}^{+\infty}\frac{1}{k!}\gamma^k\int_{\mathbb{Y}^k} f(\{u_1,\dots,u_k\})\rho(\{\gamma^{\frac{1}{d}}u_1,\dots,\gamma^{\frac{1}{d}}u_k\})\mu(\mathrm{d}u_1)\dots\mu(\mathrm{d}u_k),\end{aligned}$$

hence, the result.

A.2 Proof of Lemma 1

For all $n \in \mathbb{N}$, by independence, Φ_n is a simple point process. Moreover, for any measurable function $f : \mathbb{Y} \times N_{\mathbb{Y}} \to \mathbb{R}_+$, according to (3),

$$\begin{aligned}\mathbf{E}\Big[\sum_{y\in\Phi_n} f(y,\Phi_n\setminus\{y\})\Big] &= \sum_{i=1}^n \mathbf{E}\Big[\sum_{y\in\Phi_{n,i}} f(y,\Phi_n\setminus\{y\})\Big]\\ &= \sum_{i=1}^n \mathbf{E}\Big[\int_E f(y,\Phi_n)c_{n,i}(y,\Phi_{n,i})\mu(\mathrm{d}y)\Big]\\ &= \mathbf{E}\Big[\int_E f(y,\Phi_n)\sum_{i=1}^n c_{n,i}(y,\Phi_{n,i})\mu(\mathrm{d}y)\Big].\end{aligned}$$

B Proof of Proposition 2

By theorem 1, $d_{\mathrm{KR}}(\Phi_n,\pi_M) \le \int_{\mathbb{Y}} \mathbf{E}[|c_n(x,\Phi_n)-m(x)|]\mu(\mathrm{d}x)$.

Then, by lemma 1, $d_{\mathrm{KR}}(\Phi_n,\pi_M) \le R_n + \sum_{i=1}^n A_{n,i}$, where

$$\begin{aligned}A_{n,i} &= \int_{\mathbb{Y}} \mathbf{E}[|c_{n,i}(x,\Phi_{n,i})-\rho_{n,i}(x)|]\mu(\mathrm{d}x)\\ &= \sum_{k\ge 0}\int_{\mathbb{Y}} \mathbf{E}[|c_{n,i}(x,\Phi_{n,i})-\rho_{n,i}(x)|\mathbf{1}_{\{|\Phi_{n,i}|=k\}}]\mu(\mathrm{d}x)\\ &= B_{n,i}+C_{n,i}\end{aligned}$$

with

$$B_{n,i} = p_{n,i,0}\int_{\mathbb{Y}} |c_{n,i}(x,\varnothing)-\rho_{n,i}(x)|\mu(\mathrm{d}x),$$

$$C_{n,i} = \sum_{k\ge 1}\int_{\mathbb{Y}} \mathbf{E}[|c_{n,i}(x,\Phi_{n,i})-\rho_{n,i}(x)|\mathbf{1}_{\{|\Phi_{n,i}|=k\}}]\mu(\mathrm{d}x).$$

On one hand, by (4), for any $x\in\mathbb{Y}$, $p_{n,i,0}\rho_{n,i}(x) = p_{n,i,0}\mathbf{E}[c_{n,i}(x,\Phi_{n,i})]$, then, since $\Phi_{n,i}$ is repulsive, $p_{n,i,0}\rho_{n,i}(x) \le p_{n,i,0}c_{n,i}(x,\varnothing)$. On the other hand, still by (4), for any $x\in\mathbb{Y}$,

$$\rho_{n,i}(x) = \mathbf{E}[c_{n,i}(x,\Phi_{n,i})] \ge p_{n,i,0}c_{n,i}(x,\varnothing)$$

and it follows from both last inequalities that

$$|p_{n,i,0}c_{n,i}(x,\varnothing) - p_{n,i,0}\rho_{n,i}(x)| \le (1-p_{n,i,0})p_{n,i,0}c_{n,i}(x,\varnothing),$$

then, since by (5), $p_{n,i,0}\int_{\mathbb{Y}} c_{n,i}(x,\varnothing)\mu(\mathrm{d}x) = p_{n,i,1} \le (1-p_{n,i,0})$, we get

$$B_{n,i} \le (1-p_{n,i,0})^2.$$

Since $c_{n,i}(x,\Phi_{n,i}) \le c_{n,i}(x,\varnothing)$ and $\rho_{n,i}(x) \le c_{n,i}(x,\varnothing)$, we also have

$$
\begin{aligned}
C_{n,i} &\leq \sum_{k\geq 1} p_{n,i,k} \int_{\mathbb{Y}} c_{n,i}(x,\varnothing)\mu(\mathrm{d}x) \\
&= (1-p_{n,i,0}) \int_{\mathbb{Y}} c_{n,i}(x,\varnothing)\mu(\mathrm{d}x) \\
&\leq (1-p_{n,i,0})^2.
\end{aligned}
$$

In addition, we get from (A2) the existence of a positive constant C such that $1-p_{n,i,0} \leq C/n$, then $A_{n,i} \leq 2C^2/n^2$. Then, there exists a positive constant δ such that for any $n \in \mathbb{N}$, $A_n \leq \delta/n$. Hence, by (A1), the result.

C Proof of Proposition 3

We introduce the following notations, for all $x, y \in \mathbb{Y}$ and $k \in \mathbb{N}$:

$$
\begin{cases}
K^{(1)}(x,y) = K(x,y) \\
K^{(k+1)}(x,y) = \int_{\Lambda} K^{(k)}(x,z)K(z,y)\mu(\mathrm{d}z).
\end{cases}
$$

Hence, we obtain by recurrence,

$$
K^{(k+1)}(x,y) = \int_{\Lambda^k} K(x,z_1)K(z_1,z_2)\dots K(z_k,y)\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_k). \qquad (8)
$$

Let us now give the following technical lemma:

Lemma 2. *Suppose Φ a DPP with kernel K of local trace class and with discrete spectrum in $[0;1)$, J its associated interaction operator, and (λ_j) the associated sequence of eigenvalues of K. Then, for all $k \in \mathbb{N}$,*

$$
\int_{\mathbb{Y}} K^{(k)}(x,x)\mu(dx) = \sum_{j=1}^{+\infty} \lambda_j^k,
$$

$$
\int_{\mathbb{Y}} |J(x,x) - K(x,x)|\mu(dx) = \sum_{j=1}^{+\infty} \frac{\lambda_j^2}{1-\lambda_j}.
$$

Proof. Let us show the first equality.

By (8), for all $k \in \mathbb{N}$, $x, y \in \mathbb{Y}$,

$$
K^{(k+1)}(x,y) = \int_{\mathbb{Y}^k} K(x,z_1)K(z_1,z_2)\dots K(z_k,y)\mu(\mathrm{d}z_1)\dots\mu(\mathrm{d}z_k).
$$

Then, by (2.2), we get

$$K^{(k+1)}(x,y)=\sum_{j_1,\dots,j_{k+1}=1}^{+\infty}(\prod_{l=1}^{k+1}\lambda_{j_l})\phi_{j_1}(x)\phi_{j_{k+1}}(y)\int_{\mathbb{Y}^k}\Big(\prod_{q=1}^{k}\phi_{j_q}(z_q)\phi_{j_{q+1}}(z_q)\Big)\mu^k(\mathrm{d}z_1\dots\mathrm{d}z_k),$$

and since $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal basis,

$$K^{(k+1)}(x,y)=\sum_{j_1,\dots,j_{k+1}=1}^{+\infty}(\prod_{l=1}^{k+1}\lambda_{j_l})\phi_{j_1}(x)\phi_{j_{k+1}}(y)\mathbf{1}_{\{\lambda_{j_1}=\dots=\lambda_{j_{k+1}}\}},$$

hence, the first equality.
Moreover, by Lemma 2, for all $x\in\mathbb{Y}$,

$$J(x,x)=\sum_{k=1}^{+\infty}K^{(k)}(x,x)\geq K(x,x),$$

then,

$$\begin{aligned}\int_E|J(x,x)-K(x,x)|\mu(\mathrm{d}x)&=\sum_{k=2}^{+\infty}\int_E K^{(k)}(x,x)\mu(\mathrm{d}x)\\&=\sum_{k=2}^{+\infty}\sum_{j=1}^{+\infty}\lambda_j^k\\&=\sum_{j=1}^{+\infty}\frac{\lambda_j^2}{1-\lambda_j},\end{aligned}$$

which concludes the proof of the lemma.

By Proposition 1, $d_R(\Phi_n,\pi_M)\leq\int_\Lambda\mathbf{E}[|c_n(y,\Phi_n)-K_n(y,y)|]\mu(\mathrm{d}y)=:d_n$. As in previous proofs, we have the upper bound

$$d_n\leq U_n+V_n+R_n,$$

where

$$\begin{aligned}U_n&=\int_E\mathbf{E}\Big[|c_n(y,\Phi_n)-J_n(y,y)|\Big]\mu(\mathrm{d}y),\\V_n&=\int_E\mathbf{E}\Big[|J_n(y,y)-K_n(y,y)|\Big]\mu(\mathrm{d}y).\end{aligned}$$

By theorem 3.1. in [4], we get that $c_n(x,\xi)\leq J_n(x,x)$ for any $x\in\mathbb{R}^d$ and $\xi\in N_{\mathbb{Y}}$, which allows to write

$$U_n=\int_\Lambda\mathbf{E}\Big[J_n(y,y)-c_n(y,\Phi_n)\Big]\mu(\mathrm{d}y),$$

then, since by equation (4), for all $y \in \mathbb{Y}$,

$$K_n(y,y) = \mathbf{E}\Big[c_n(y,\Phi_n)\Big]$$

we get

$$U_n = \int_E J_n(y,y) - K_n(y,y)\mu(\mathrm{d}y).$$

By Lemma 2, for all $n \in \mathbb{N}$,

$$\int_E J_n(y,y) - K_n(y,y)\mu(\mathrm{d}y) = \sum_{j=1}^{+\infty} \frac{\lambda_{n,j}^2}{1-\lambda_{n,j}}$$

where the $(\lambda_{n,j})_{j\geq 1}$ are the eigenvalues of K_n.

If for all $x, y \in \mathbb{Y}$, $K(x,y) = \sum_{j=1}^{+\infty} \lambda_j \phi_j(x)\phi_j(y)$, then for all $x, y \in \Lambda$,

$$\begin{aligned} K_n(x,y) &= \sum_{j=1}^{+\infty} \lambda_j \phi_j(\frac{x}{\sqrt{\beta_n}})\phi_j(\frac{y}{\sqrt{\beta_n}}) \\ &= \sum_{j=1}^{+\infty} \lambda_j N_{n,j} \widetilde{\phi}_j(x)\widetilde{\phi}_j(y), \end{aligned}$$

where for all $j \in \mathbb{N}$ and all $x \in \mathbb{Y}$,

$$\widetilde{\phi}_j(x) = \frac{1}{\sqrt{N_{n,j}}}\phi_j(\frac{x}{\sqrt{\beta_n}})$$

and for all $j \in \mathbb{N}$,

$$N_{n,j} = \sqrt{\beta_n}\int_E |\phi_j(u)|^2 \mu(du) \leq \sqrt{\beta_n},$$

such that $(\widetilde{\phi}_j)_{j\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{Y},\mu;\mathbb{C})$.

Then $\lambda_{n,j} = \lambda_j N_{n,j}$ and we can deduce that there exists a $\theta > 0$ independent of Λ such that for all $n \in \mathbb{N}$,

$$U_n \leq \theta\beta_n,$$

as expected.

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