Determinantal point processes

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August 11, 2015

Abstract

In this survey we review two topics concerning determinantal (or fermion) point processes. First, we provide the construction of diffusion processes on the space of configurations whose invariant measure is the law of a determinantal point process. Second, we present some algorithms to sample from the law of a determinantal point process on a finite window. Related open problems are listed.

Keywords: Determinantal point processes; diffusions processes; Dirichlet forms; stochastic simulation.

Mathematics Subject Classification: 60G55, 60H07.

1   Introduction

Determinantal (or fermion) point processes have been introduced in [27] to represent configurations of fermions. Determinantal point processes play a fundamental role in the theory of random matrices as the eigenvalues of many ensembles of random matrices form a determinantal point process, see e.g. [18]. The full existence theorem for these processes was proved in [34], in which many examples occurring in mathematics and physics were discussed. The construction of [34] has been extended in [32] with the introduction of the family of α-determinantal point processes.

Determinantal point processes have notable mathematical properties, e.g. their Laplace transforms, Janossy densities and Papangelou conditional intensities admit closed form expressions. Due to their repulsive nature, determinantal point processes have been recently proposed as models for nodes' locations in wireless communication, see [28] and [35].

This paper is structured as follows. In Section 2 we give some preliminaries on point processes, including the definition of determinantal point processes,

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the expression of their Laplace transform (Theorem 2.3), Janossy densities (Proposition 2.4) and Papangelou intensity (Theorems 2.6 and 2.7), cf. [9], [10], [16], [16], [29], [23], [18] and [32]. We also refer to [7], [13] and [33] for the required background on functional analysis.

In Section 3 we review the integration by parts formula for determinantal point processes and its extension by closability, cf. [8] and [11].

In Section 4 we report a result in [11] on the construction of a diffusion on the space of configurations which has the law of a determinantal point process as invariant measure. To this aim we use arguments based on the theory of Dirichlet forms, cf. [14], [25] and the Appendix. It has to be noticed that the construction of the diffusion provided in [11] differs from that one given in [37], where alternative techniques are used.

Section 5 deals with the simulation of determinantal point processes. We provide two different simulation algorithms to sample from the law of a determinantal point processes on a compact. In particular, we describe the (standard) sampling algorithm given in [17] (see Algorithm 1 below) and an alternative simulation algorithm obtained by specializing the well-known routine to sample from the law of a finite point process with bounded Papangelou conditional intensity (see e.g. [19], [20], [23] and Algorithm 2 below). We show that the number of steps in the latter algorithm grows logarithmically with the size of the initial dominating point process, which gives a rough idea of the simulation time required by this algorithm. Finally, we propose a new approximate simulation algorithm for the Ginibre point process, which presents advantages in terms of complexity and CPU time.

Finally, some open problems are listed in Section 6.

2 Preliminaries

Locally finite point processes, correlation functions, Janossy density and Papangelou intensity

Let $\mathbb{X}$ be a locally compact second countable Hausdorff space, and $\mathscr{X}$ be the Borel σ-algebra on $\mathbb{X}$. For any subset $A \subseteq \mathbb{X}$, let $|A|$ denote the cardinality of A, setting $|A| = \infty$ if A is not finite. We denote by $\mathbf{N}_\sigma$ the set of locally finite point configurations on $\mathbb{X}$:

$$\mathbf{N}_\sigma := \{\xi \subseteq \mathbb{X} \; : \; |\xi \cap \Lambda| < \infty \quad \text{for all relatively compact sets } \Lambda \subset \mathbb{X}\}.$$

In fact, $\mathbf{N}_\sigma$ can be identified with the set of all simple nonnegative integer-valued Radon measures on $\mathbb{X}$ (an integer-valued Radon measure ν is said to be simple if for all $x \in \mathbb{X}$, $\nu(\{x\}) \in \{0, 1\}$). Hence, it is naturally topologized by the vague topology, which is the weakest topology such that for any continuous and compactly supported function f on $\mathbb{X}$, the mapping

$$\xi \mapsto \langle f, \xi \rangle := \sum_{y \in \xi} f(y)$$

is continuous. We denote by $\mathscr{N}_\sigma$ the corresponding Borel σ-field. For $\xi \in \mathbf{N}_\sigma$, we write $\xi \cup y_0 = \xi \cup \{y_0\}$ for the addition of a particle at y_0 and $\xi \setminus y_0 = \xi \setminus \{y_0\}$ for the removal of a particle at y_0. We define the set of finite point configurations on $\mathbb{X}$ by

$$\mathbf{N}_\sigma^f := \{\xi \subseteq \mathbb{X} : |\xi| < \infty\},$$

which is equipped with the trace σ-algebra $\mathscr{N}_\sigma^f = \mathscr{N}_\sigma|_{\mathbf{N}_\sigma^f}$. For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, let $\mathbf{N}_\sigma(\Lambda)$ be the space of finite configurations on Λ, and $\mathscr{N}_\sigma(\Lambda)$ the associated (trace-) σ-algebra. As in [16], we define for any Radon measure μ on $\mathbb{X}$ the (μ-)sample measure L^μ on $(\mathbf{N}_\sigma^f, \mathscr{N}_\sigma^f)$ by

$$\int_{\mathbf{N}_\sigma^f} f(\alpha) \, L^\mu(\mathrm{d}\alpha) := \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{X}^n} f(\{x_1, \ldots, x_n\}) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n), \qquad (2.1)$$

for any measurable $f : \mathbf{N}_\sigma^f \to \mathbb{R}_+$. Similarly, we define its restriction to the relatively compact set $\Lambda \subseteq \mathbb{X}$ by

$$\int_{\mathbf{N}_\sigma(\Lambda)} f(\alpha) \, L_\Lambda^\nu(\mathrm{d}\alpha) := \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\}) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n),$$

for any measurable $f : \mathbf{N}_\sigma(\Lambda) \to \mathbb{R}_+$. A simple and locally finite point process η is defined as a random element on a probability space $(\Omega, \mathcal{A})$ with values in $\mathbf{N}_\sigma$. We denote its distribution by $\mathbb{P}$. It is characterized by its Laplace transform $\mathcal{L}_\eta$, which is defined, for any measurable nonnegative function f on $\mathbb{X}$, by

$$\mathcal{L}_\eta(f) = \int_{\mathbf{N}_\sigma} e^{-\langle f, \xi \rangle} \, \mathbb{P}(\mathrm{d}\xi).$$

We denote the expectation of an integrable random variable F defined on $(\mathbf{N}_\sigma, \mathscr{N}_\sigma, \mathbb{P})$ by

$$\mathbb{E}[F(\eta)] := \int_{\mathbf{N}_\sigma} F(\xi) \, \mathbb{P}(\mathrm{d}\xi).$$

For ease of notation, we define by

$$\xi_A := \xi \cap A,$$

the restriction of $\xi \in \mathbf{N}_\sigma$ to a set $A \subset \mathbb{X}$. The restriction of $\mathbb{P}$ to $\mathscr{N}_\sigma(A)$ is denoted by $\mathbb{P}_A$ and the number of points of ξ_A, i.e. $\xi(A) := |\xi \cap A|$, is denoted by $\xi(A)$. A point process η is said to have a correlation function $\rho : \mathbf{N}_\sigma^f \to [0, \infty)$ with respect to (w.r.t.) a Radon measure μ on $(\mathbb{X}, \mathscr{X})$ if ρ is measurable and

$$\int \sum_{\alpha \subset \xi, \ \alpha \in \mathbf{N}_\sigma^f} f(\alpha) \, \mathbb{P}(\mathrm{d}\xi) = \int_{\mathbf{N}_\sigma^f} f(\alpha) \, \rho(\alpha) \, L^\mu(\mathrm{d}\alpha),$$

for all measurable nonnegative functions f on $\mathbf{N}_\sigma^f$. When such a measure μ exists, it is known as the intensity measure of η. For $\alpha = \{x_1, \ldots, x_k\}$, where $k \geq 1$, we will sometimes write $\rho(\alpha) = \rho_k(x_1, \ldots, x_k)$ and call ρ_k the k-th correlation function. Here ρ_k is a symmetric function on $\mathbb{X}^k$. Similarly, the correlation functions of η, w.r.t. a Radon measure μ on $\mathbb{X}$, are (if they exist) measurable symmetric functions $\rho_k : \mathbb{X}^k \longrightarrow [0, \infty)$ such that

$$\mathbb{E}\left[\prod_{i=1}^k \eta(B_i)\right] = \int_{B_1 \times \ldots \times B_k} \rho_k(x_1, \ldots, x_k) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_k),$$

for any family of mutually disjoint bounded subsets $B_1, \ldots, B_k$ of $\mathbb{X}$, $k \geq 1$. The previous formula can be generalized as follows:

Proposition 2.1 *Let $B_1, \ldots, B_n$ be disjoint bounded Borel subsets of $\mathbb{X}$. Let $k_1, \ldots, k_n$ be integers such that $\sum_{i=1}^{n} k_i = N$. Then,*

$$\mathbb{E}\left[\prod_{i=1}^{n} \frac{\eta(B_i)!}{(\eta(B_i) - k_i)!}\right] = \int_{B_1^{k_1} \times \ldots \times B_n^{k_n}} \rho(\{x_1, \ldots, x_N\}) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_N).$$

We require in addition that $\rho_n(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$ for some $1 \leq i \neq j \leq n$. Heuristically, ρ_1 is the particle density with respect to μ, and

$$\rho_n(x_1, \ldots, x_n) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n)$$

is the probability of finding a particle in the vicinity of each x_i, $i = 1, \ldots, n$. For any relatively compact subset $\Lambda \subseteq \mathbb{X}$, the Janossy densities of η, *w.r.t.* a Radon measure μ on $\mathbb{X}$, are (if they exist) measurable functions $j_\Lambda^n : \Lambda^n \to [0, \infty)$ satisfying for all measurable functions $f : \mathbf{N}_\sigma(\Lambda) \to [0, \infty)$,

$$\mathbb{E}\left[f(\eta_\Lambda)\right] = \sum_{n \geq 0} \frac{1}{n!} \int_{\Lambda^n} f(\{x_1, \ldots, x_n\}) \, j_\Lambda^n(x_1, \cdots, x_n) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n). \tag{2.2}$$

Using the simplified notation $j_\Lambda(\alpha) := j_\Lambda^n(x_1, \ldots, x_n)$, for $\alpha = \{x_1, \ldots, x_n\}$, where $n \geq 1$, by (2.2) it follows that j_Λ is the density of $\mathbb{P}_\Lambda$ with respect to L_Λ^μ, when $\mathbb{P}_\Lambda \ll L_\Lambda^\mu$. Now we list some properties of the Janossy densities.

- Symmetry:
$$j_\Lambda^n\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) = j_\Lambda^n\left(x_1, \ldots, x_n\right),$$

  for every permutation σ of $\{1, \ldots, n\}$.

- Normalization constraint: for each relatively compact subset $\Lambda \subseteq \mathbb{X}$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} j_\Lambda^n(x_1, \ldots, x_n) \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n) = 1.$$

For $n \geq 1$, the Janossy density $j_\Lambda^n(x_1, \ldots, x_n)$ is in fact the joint density (multiplied by a constant) of the n points given that the point process has exactly n points. For $n = 0$, $j_\Lambda^0(\emptyset)$ is the probability that there are no points in Λ. We also recall that the Janossy densities can be recovered from the correlation functions via the relation

$$j_\Lambda^n(x_1, \ldots, x_n) = \sum_{m \geq 0} \frac{(-1)^m}{m!} \int_{\Lambda^m} \rho_{n+m}(x_1, \ldots, x_n, y_1, \ldots, y_m) \, \mu(\mathrm{d}y_1) \cdots \mu(\mathrm{d}y_m),$$

and vice-versa using the equality

$$\rho_n(x_1, \ldots, x_n) = \sum_{m \geq 0} \frac{1}{m!} \int_{\Lambda^m} j_\Lambda^{m+n}(x_1, \ldots, x_n, y_1, \ldots, y_m) \, \mu(\mathrm{d}y_1) \cdots \mu(\mathrm{d}y_m),$$

see [9, Theorem 5.4.II].

4

Following [16], we now recall the definition of the so-called reduced and reduced compound Campbell measures. The reduced Campbell measure of a point process η is the measure C_η on the product space $(\mathbb{X} \times \mathbf{N}_\sigma, \mathscr{X} \otimes \mathscr{N}_\sigma)$ defined by

$$C_\eta(A \times B) = \int_{\mathbf{N}_\sigma} \sum_{x \in \xi} \mathbf{1}_A(x) \mathbf{1}_B(\xi \setminus x) \, \mathbb{P}(\mathrm{d}\xi).$$

The reduced compound Campbell measure of a point process η is the measure $\hat{C}_\eta$ on the product space $(\mathbf{N}_\sigma^f \times \mathbf{N}_\sigma, \mathscr{N}_\sigma^f \otimes \mathscr{N}_\sigma)$ defined by

$$\hat{C}_\eta(A \times B) = \int_{\mathbf{N}_\sigma} \sum_{\alpha \subset \xi, \ \alpha \in \mathbf{N}_\sigma^f} \mathbf{1}_A(\alpha) \mathbf{1}_B(\xi \setminus \alpha) \, \mathbb{P}(\mathrm{d}\xi).$$

The integral versions of the equations above can be written respectively as

$$\int h(x, \xi) \, C_\eta(\mathrm{d}x \times \mathrm{d}\xi) = \int \sum_{x \in \xi} h(x, \xi \setminus x) \, \mathbb{P}(\mathrm{d}\xi), \tag{2.3}$$

for all nonnegative measurable functions $h : \mathbb{X} \times \mathbf{N}_\sigma \to \mathbb{R}_+$, and

$$\int h(\alpha, \xi) \, \hat{C}_\eta(\mathrm{d}\alpha \times \mathrm{d}\xi) = \int \sum_{\alpha \subset \xi, \ \alpha \in \mathbf{N}_\sigma^f} h(\alpha, \xi \setminus \alpha) \, \mathbb{P}(\mathrm{d}\xi),$$

for all nonnegative measurable functions $h : \mathbf{N}_\sigma^f \times \mathbf{N}_\sigma \to \mathbb{R}_+$. Comparing (2.3) with the well-known Mecke formula (see (7) in [21]) leads us to introduce the following condition:

(Σ): $C_\eta \ll \mu \otimes \mathbb{P}$.

The Radon-Nikodym derivative c of C_η w.r.t. $\mu \otimes \mathbb{P}$ is called (a version of) the Papangelou intensity of η. Assumption (Σ) implies that $\hat{C}_\eta \ll L^\mu \otimes \mathbb{P}$ and we denote the Radon-Nikodym derivative of $\hat{C}_\eta$ w.r.t. $L^\mu \otimes \mathbb{P}$ by $\hat{c}$, and call $\hat{c}$ the compound Papangelou intensity of η. One then has for any $\xi \in \mathbf{N}_\sigma$, $\hat{c}(\emptyset, \xi) = 1$, as well as for all $x \in \mathbb{X}$, $\hat{c}(x, \xi) = c(x, \xi)$. The Papangelou intensity c has the following interpretation:

$$c(x, \xi) \, \mu(\mathrm{d}x)$$

is the probability of finding a particle in the vicinity of $x \in \mathbb{X}$ conditional on the configuration ξ.

The compound Papangelou intensity verifies the following commutation relation:

$$\hat{c}(\nu, \eta \cup \xi) \, \hat{c}(\eta, \xi) = \hat{c}(\nu \cup \eta, \xi), \tag{2.4}$$

for all $\eta, \nu \in \mathbf{N}_\sigma^f$ and $\xi \in \mathbf{N}_\sigma$. The recursive application of the previous relation also yields

$$\hat{c}(\{x_1, \ldots, x_n\}, \xi) = \prod_{k=1}^{n} c(x_k, \xi \cup x_1 \cup \cdots \cup x_{k-1}),$$

5

for all $x_1, \ldots, x_n \in \mathbb{X}$ and $\xi \in \mathbf{N}_\sigma$, where we have used the convention $x_0 := \emptyset$.

The assumption (Σ), along with the definition of the reduced Campbell measure, allows us to write the following identity, known as the Georgii-Nguyen-Zessin identity:

$$\int_{\mathbf{N}_\sigma} \sum_{y \in \xi} u(y, \xi \setminus y) \, \mathbb{P}(\mathrm{d}\xi) = \int_{\mathbf{N}_\sigma} \int_{\mathbb{X}} u(z, \xi) \, c(z, \xi) \, \mu(\mathrm{d}z) \mathbb{P}(\mathrm{d}\xi), \qquad (2.5)$$

for all measurable nonnegative functions $u : \mathbb{X} \times \mathbf{N}_\sigma \to \mathbb{R}_+$. We also have a similar identity for the compound Papangelou intensity:

$$\int_{\mathbf{N}_\sigma} \sum_{\alpha \subset \xi, \ \alpha \in \mathbf{N}_\sigma^f} u(\alpha, \xi \setminus \alpha) \, \mathbb{P}(\mathrm{d}\xi) = \int_{\mathbf{N}_\sigma} \int_{\mathbf{N}_\sigma^f} u(\alpha, \xi) \, \hat{c}(\alpha, \xi) \, L^\mu(\mathrm{d}\alpha) \mathbb{P}(\mathrm{d}\xi), \quad (2.6)$$

for all measurable functions $u : \mathbf{N}_\sigma^f \times \mathbf{N}_\sigma \to \mathbb{R}_+$.

Note that equations (2.5) and (2.6) are generalizations of equations (7) and (8) of [21]. Indeed, in the case of the Poisson point process, $c(z, \xi) = 1$ and $c(\alpha, \xi) = 1$.

Combining relation (2.5) and the definition of the correlation functions, we find

$$\mathbb{E}[c(x, \eta)] = \rho_1(x),$$

for μ-a.e. $x \in \mathbb{X}$. More generally, using (2.6), we also have

$$\mathbb{E}[\hat{c}(\alpha, \eta)] = \rho(\alpha), \qquad (2.7)$$

for $\mathbb{P}$-a.e. $\alpha \in \mathbf{N}_\sigma^f$.

Kernels and integral operators

As usual, we denote by $\mathbb{X}$ a locally compact second countable Hausdorff space and by μ a Radon measure on $\mathbb{X}$. For any compact set $\Lambda \subseteq \mathbb{X}$, we denote by $L^2(\Lambda, \mu)$ the Hilbert space of complex-valued square integrable functions $w.r.t.$ the restriction of the Radon measure μ on Λ, equipped with the inner product

$$\langle f, g \rangle_{L^2(\Lambda, \mu)} := \int_\Lambda f(x) \overline{g(x)} \, \mu(\mathrm{d}x), \quad f, g \in L^2(\Lambda, \mu),$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. By definition, an integral operator $\mathcal{K} : L^2(\mathbb{X}, \mu) \to L^2(\mathbb{X}, \mu)$ with kernel $K : \mathbb{X}^2 \to \mathbb{C}$ is a bounded operator defined by

$$\mathcal{K}f(x) := \int_{\mathbb{X}} K(x, y) f(y) \, \mu(\mathrm{d}y), \quad \text{for } \mu\text{-almost all } x \in \mathbb{X}.$$

We denote by $\mathcal{P}_\Lambda$ the projection operator from $L^2(\mathbb{X}, \mu)$ to $L^2(\Lambda, \mu)$ and define the operator $\mathcal{K}_\Lambda = \mathcal{P}_\Lambda \mathcal{K} \mathcal{P}_\Lambda$. We note that the kernel of $\mathcal{K}_\Lambda$ is given by $K_\Lambda(x, y) := \mathbf{1}_\Lambda(x) K(x, y) \mathbf{1}_\Lambda(y)$, for $x, y \in \mathbb{X}$. It can be shown that $\mathcal{K}_\Lambda$ is a compact operator.

The operator $\mathcal{K}$ is said to be Hermitian or self-adjoint if its kernel verifies

$$K(x,y) = \overline{K(y,x)}, \quad \text{for } \mu^{\otimes 2}\text{-almost all } (x,y) \in \mathbb{X}^2. \tag{2.8}$$

Equivalently, this means that the integral operators $\mathcal{K}_\Lambda$ are self-adjoint for any compact set $\Lambda \subseteq \mathbb{X}$. If $\mathcal{K}_\Lambda$ is self-adjoint, by the spectral theorem for self-adjoint and compact operators we have that $L^2(\Lambda, \mu)$ has an orthonormal basis $\{\varphi_j^\Lambda\}_{j \geq 1}$ of eigenfunctions of $\mathcal{K}_\Lambda$. The corresponding eigenvalues $\{\mu_j^\Lambda\}_{j \geq 1}$ have finite multiplicity (except possibly the zero eigenvalue) and the only possible accumulation point of the eigenvalues is the zero eigenvalue. In that case, the kernel K_Λ of $\mathcal{K}_\Lambda$ can be written as

$$K_\Lambda(x,y) = \sum_{n \geq 1} \mu_n^\Lambda \varphi_n^\Lambda(x) \overline{\varphi_n^\Lambda(y)}, \tag{2.9}$$

for $x, y \in \Lambda$. We say that an operator $\mathcal{K}$ is positive (respectively non-negative) if its spectrum is included in $(0, +\infty)$ (respectively $[0, +\infty)$). For two operators $\mathcal{K}$ and $\mathcal{I}$, we say that $\mathcal{K} > \mathcal{I}$ (respectively $\mathcal{K} \geq \mathcal{I}$) in the operator ordering if $\mathcal{K} - \mathcal{I}$ is a positive operator (respectively non-negative operator).

We say that a self-adjoint integral operator $\mathcal{K}_\Lambda$ is of trace class if

$$\sum_{n \geq 1} |\mu_n^\Lambda| < \infty,$$

and define the trace of $\mathcal{K}_\Lambda$ as $\operatorname{Tr} \mathcal{K}_\Lambda = \sum_{n \geq 1} \mu_n^\Lambda$. If $\mathcal{K}_\Lambda$ is of trace class for every compact subset $\Lambda \subseteq \mathbb{X}$, then we say that $\mathcal{K}$ is locally of trace class. It is easily seen that if a Hermitian integral operator $\mathcal{K} : L^2(\mathbb{X}, \mu) \to L^2(\mathbb{X}, \mu)$ is of trace class, then $\mathcal{K}^n$ is also of trace class for all $n \geq 2$. Indeed, $\operatorname{Tr}(\mathcal{K}^n) \leq \|\mathcal{K}\|_{op}^{n-1} \operatorname{Tr}(\mathcal{K})$, where $\|\mathcal{K}\|_{op}$ is the operator norm of $\mathcal{K}$.

Let Id denote the identity operator on $L^2(\mathbb{X}, \mu)$ and let $\mathcal{K}$ be a trace class operator on $L^2(\mathbb{X}, \mu)$. We define the Fredholm determinant of $\text{Id} + \mathcal{K}$ as

$$\operatorname{Det}(\text{Id} + \mathcal{K}) = \exp\left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \operatorname{Tr}(\mathcal{K}^n)\right). \tag{2.10}$$

It turns out that

$$\operatorname{Det}(\text{Id} + \mathcal{K}) = \sum_{n \geq 0} \frac{1}{n!} \int_{\mathbb{X}^n} \det(K(x_i, x_j))_{1 \leq i,j \leq n} \, \mu(dx_1) \cdots \mu(dx_n), \tag{2.11}$$

where K is the kernel of $\mathcal{K}$ and $\det(K(x_i, x_j))_{1 \leq i,j \leq n}$ is the determinant of the $n \times n$ matrix $(K(x_i, x_j))_{1 \leq i,j \leq n}$. Equation (2.11) was obtained in Theorem 2.4 of [32], see also [7] for more details on the Fredholm determinant.

We end this section by recalling the following result from [16, Lemma A.4]:

Proposition 2.2 *Let $\mathcal{K}$ be a non-negative and locally of trace class integral operator on $L^2(\mathbb{X}, \mu)$. Then one can choose its kernel K (defined everywhere) such that the following properties hold:*

(i) K is non-negative, in the sense that for any $c_1, \ldots, c_n \in \mathbb{C}$ and μ-a.e. $x_1, \ldots, x_n \in \mathbb{X}$, we have $\sum_{i,j=1}^{n} \bar{c}_i K(x_i, x_j) c_j \geq 0$.

(ii) K is a Carleman kernel, i.e. $K_x = K(\cdot, x) \in L^2(\mathbb{X}, \mu)$ for μ-a.e. $x \in \mathbb{X}$.

(iii) For any compact subset $\Lambda \subseteq \mathbb{X}$, $\operatorname{Tr} \mathcal{K}_\Lambda = \int_\Lambda K(x, x)\, \mu(\mathrm{d}x)$ and

$$\operatorname{Tr} \left(\mathcal{P}_\Lambda \mathcal{K}^k \mathcal{P}_\Lambda \right) = \int_\Lambda \langle K_x, \mathcal{K}^{k-2} K_x \rangle_{L^2(\Lambda, \mu)}\, \mu(\mathrm{d}x),$$

for $k \geq 2$.

Henceforth, the kernel of a non-negative and locally of trace class integral operator $\mathcal{K}$ will be chosen according to the previous proposition.

Determinantal point processes

A locally finite and simple point process η on $\mathbb{X}$ is called determinantal point process if its correlation functions *w.r.t.* the Radon measure μ on $(\mathbb{X}, \mathcal{X})$ exist and are of the form

$$\rho_k(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i, j \leq k},$$

for any $k \geq 1$ and $x_1, \ldots, x_k \in \mathbb{X}$, where $K(\cdot, \cdot)$ is a measurable function. Throughout this paper we shall consider the following hypothesis:

(H1): *The operator $\mathcal{K}$ is locally of trace class, satisfies (2.8), and its spectrum is contained in $[0, 1)$, i.e. $0 \leq \mathcal{K} < \operatorname{Id}$ in the operator ordering. We denote by K the kernel of $\mathcal{K}$.*

By the results in [27] and [34] (see also Lemma 4.2.6 and Theorem 4.5.5 in [18]), it follows that under **(H1)**, there exists a unique (in law) determinantal point process with integral operator $\mathcal{K}$. In this survey, we shall only consider determinantal point processes with Hermitian kernel. However, we mention that many important examples of determinantal point processes exhibit a non-Hermitian kernel, see [2]-[6], [24] and [36].

Let us now recall the following result from e.g. [32] (see Theorem 3.6 therein) that gives the Laplace transform of η.

Theorem 2.3 *Let $\mathcal{K}$ be an operator satisfying (**H1**) and η the determinantal point process with kernel K. Then η has Laplace transform*

$$\mathcal{L}_\eta(f) = \operatorname{Det}\left(\operatorname{Id} - \mathcal{K}_\varphi \right),$$

for each nonnegative f on $\mathbb{X}$ with compact support, where $\varphi = 1 - e^{-f}$ and $\mathcal{K}_\varphi$ is the trace class integral operator with kernel

$$K_\varphi(x, y) = \sqrt{\varphi(x)} K(x, y) \sqrt{\varphi(y)}, \quad x, y \in \mathbb{X}.$$

Let $\mathcal{K}$ be an operator satisfying assumption (**H1**). We define the operators on $L^2(\mathbb{X}, \mu)$:

$$\mathcal{J} := (\operatorname{Id} - \mathcal{K})^{-1} \mathcal{K}, \tag{2.12}$$

and

$$\mathcal{J}[\Lambda] := (\mathrm{Id} - \mathcal{K}_\Lambda)^{-1}\mathcal{K}_\Lambda, \tag{2.13}$$

where Λ is a compact subset of $\mathbb{X}$. The operator $\mathcal{J}$ is called global interaction operator and the operator $\mathcal{J}[\Lambda]$ is called local interaction operator. We emphasize that, unlike $\mathcal{K}_\Lambda$, $\mathcal{J}[\Lambda]$ is not a projection operator, i.e. in general $\mathcal{J}[\Lambda] \neq \mathcal{P}_\Lambda \mathcal{J} \mathcal{P}_\Lambda$. In any case, $\mathcal{J}[\Lambda]$ has some notable properties, as proved in [16]. First, it is easily seen that $\mathcal{J}[\Lambda]$ exists as a bounded operator and its spectrum is included in $[0, +\infty)$. Second, $\mathcal{J}[\Lambda]$ is also an integral operator, and we denote by $J[\Lambda]$ its kernel (in fact, one can even show that $\mathcal{J}[\Lambda]$ is a Carleman operator, cfr. the beginning of Section 3 in [16]). Third, $\mathcal{J}[\Lambda]$ is a trace class operator. Finally, by (2.9) we have

$$J[\Lambda](x,y) = \sum_{n \geq 1} \frac{\mu_n^\Lambda}{1 - \mu_n^\Lambda} \varphi_n^\Lambda(x)\overline{\varphi_n^\Lambda(y)},$$

for $x, y \in \Lambda$.

For $\alpha = \{x_1, \ldots, x_k\} \in \mathbf{N}_\sigma(\Lambda)$, we denote by $\det J[\Lambda](\alpha)$ the determinant $\det(J[\Lambda](x_i, x_j))_{1 \leq i,j \leq k}$. Note that for all $k \in \mathbb{N}^*$, the function

$$(x_1, \ldots, x_k) \mapsto \det J[\Lambda](\{x_1, \ldots, x_k\})$$

is $\mu^{\otimes k}$-a.e. non-negative (thanks to Proposition 2.2) and symmetric in $x_1, \ldots, x_k$ (see e.g. the Appendix of [16]), and we simply write $\det J[\Lambda](\{x_1, \ldots, x_k\}) = \det J[\Lambda](x_1, \ldots, x_k)$. The relevance of the local interaction operator becomes clear when computing the Janossy densities of the determinantal point process. More precisely, the following proposition holds.

Proposition 2.4 (Lemma 3.3 of [32]) *Let $\mathcal{K}$ be an operator satisfying* (**H1**) *and η the determinantal point process with kernel K. Then, for a compact subset $\Lambda \subseteq \mathbb{X}$ and $n \in \mathbb{N}^*$, the determinantal process η admits Janossy densities*

$$j_\Lambda^n(x_1, \ldots, x_n) = \mathrm{Det}(\mathrm{Id} - \mathcal{K}_\Lambda) \det J[\Lambda](x_1, \ldots, x_k), \tag{2.14}$$

for $x_1, \ldots, x_k \in \Lambda$. The void probability is equal to $j_\Lambda^0(\emptyset) = \mathrm{Det}(\mathrm{Id} - \mathcal{K}_\Lambda)$.

We emphasize that (2.14) still makes sense if $\|\mathcal{K}_\Lambda\|_{op} = 1$; indeed the zeros of $\mathrm{Det}(\mathrm{Id} - \mathcal{K}_\Lambda)$ are of the same order of the poles of $\det J[\Lambda](x_1, \ldots, x_k)$, see Lemma 3.4 of [32] for a more formal proof.

We now give some properties linking the rank of $\mathcal{K}$, $\mathrm{Rank}(\mathcal{K})$, and the number of points of the determinantal point process with integral operator $\mathcal{K}$.

Proposition 2.5 (Theorem 4 in [34], see also [18]) *Let $\mathcal{K}$ be an operator satisfying* (**H1**) *and η the determinantal point process with kernel K. We have:*

a) *The probability of the event that the number of points is finite is either 0 or 1, depending on whether $\mathrm{Tr}(\mathcal{K})$ is finite or infinite. The number of points in a compact subset $\Lambda \subseteq \mathbb{X}$ is finite since $\mathrm{Tr}(\mathcal{K}_\Lambda) < \infty$.*

b) *The number of points is less than or equal to $n \in \mathbb{N}^*$ with probability 1 if and only if $\mathcal{K}$ is a finite rank operator satisfying $\mathrm{Rank}(\mathcal{K}) \leq n$.*

9

c) *The number of points is $n \in \mathbb{N}^*$ with probability 1 if and only if $\mathcal{K}$ is an orthogonal projection satisfying* $\mathrm{Rank}(\mathcal{K}) = n$.

We now give the Papangelou intensity of determinantal point processes.

Theorem 2.6 (Theorem 3.1 of [16]) *Let $\mathcal{K}$ be an operator satisfying* (**H1**) *and η the determinantal point process with kernel K. Then, for each compact set $\Lambda \subseteq \mathbb{X}$, η_Λ satisfies condition* (Σ) *(with μ_Λ in place of μ). A version of its compound Papangelou intensity $\hat{c}_\Lambda$ is given by*

$$\hat{c}_\Lambda(\alpha, \xi) = \frac{\det J[\Lambda](\alpha \cup \xi)}{\det J[\Lambda](\xi)}, \quad \alpha \in \mathbf{N}_\sigma^f, \ \xi \in \mathbf{N}_\sigma,$$

where the ratio is defined to be zero whenever the denominator vanishes. This version also satisfies the inequalities

$$\hat{c}_\Lambda(\alpha, \xi) \geq \hat{c}_\Lambda(\alpha, \xi'), \quad and \quad 0 \leq \hat{c}_\Lambda(\alpha, \xi) \leq \det J[\Lambda](\alpha) \leq \prod_{x \in \alpha} J[\Lambda](x, x),$$

$$(2.15)$$

whenever $\xi \subset \xi' \in \mathbf{N}_\sigma(\Lambda)$ and $\alpha \in \mathbf{N}_\sigma(\Lambda) \setminus \omega$.

Let $\mathcal{K}$ be an operator satisfying (**H1**) and let η be the determinantal point process with kernel K. Let $\mathcal{J}$ be the operator defined in (2.12). As proved in [16], $\mathcal{J}$ satisfies the following properties: it is locally of trace class and its kernel $(x, y) \mapsto J(x, y)$ can be chosen to satisfy Proposition 2.2. Moreover, η is stochastically dominated by a Poisson point process with mean measure $J(x, x)\, \mu(\mathrm{d}x)$ i.e., denoting by $\widetilde{\mathbb{P}}$ the law of the Poisson process,

$$\int f\, \mathrm{d}\mathbb{P} \leq \int f\, \mathrm{d}\widetilde{\mathbb{P}},$$

for all increasing measurable f. Here, we say that f is increasing if $f(\xi) \leq f(\xi')$ whenever $\xi \subset \xi' \in \mathbf{N}_\sigma$.

We finally report the following theorem.

Theorem 2.7 (Theorem 3.6 in [16]) *Let $\mathcal{K}$ be an operator satisfying* (**H1**) *and η the determinantal point process with kernel K. Then η satisfies condition* (Σ), *and its compound Papangelou intensity is given by*

$$\hat{c}(\alpha, \xi) = \lim_{n \to \infty} \hat{c}_{\Delta_n}(\alpha, \xi_{\Delta_n}), \quad for\ L^\mu \otimes \mathbb{P} \text{ - almost every } (\alpha, \xi), \qquad (2.16)$$

where $(\Delta_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact sets in $\mathbb{X}$ converging to $\mathbb{X}$.

In general (2.16) does not give a closed form for the compound Papangelou intensity. In order to write $\hat{c}$ in closed form, additional hypotheses have to be assumed, see Proposition 3.9 in [16].

3 Integration by parts

Hereafter we assume that $\mathbb{X}$ is a subset of $\mathbb{R}^d$, equipped with the Euclidean distance, μ is a Radon measure on $\mathbb{X}$ and $\Lambda \subseteq \mathbb{X}$ is a fixed compact set. We denote by $x^{(i)}$ the ith component of $x \in \mathbb{R}^d$.

Differential calculus

We denote by $\mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d)$ the set of all $\mathcal{C}^\infty$-vector fields $v : \Lambda \longrightarrow \mathbb{R}^d$ (with compact support) and by $\mathcal{C}_b^\infty(\Lambda^k)$ the set of all $\mathcal{C}^\infty$-functions on Λ^k whose derivatives are bounded.

Definition 1 *A function $F : \mathbf{N}_\sigma(\Lambda) \to \mathbb{R}$ is said to be in $\mathscr{S}_\Lambda$ if*

$$F(\xi_\Lambda) = f_0 \mathbf{1}_{\{\xi(\Lambda)=0\}} + \sum_{k=1}^n \mathbf{1}_{\{\xi(\Lambda)=k\}} f_k(\xi_\Lambda), \tag{3.1}$$

for some integer $n \geq 1$, where for $k = 1, \ldots, n$, $f_k \in \mathcal{C}_b^\infty(\Lambda^k)$ is a symmetric function and $f_0 \in \mathbb{R}$ is a constant.

The gradient of $F \in \mathscr{S}_\Lambda$ of the form (3.1) is defined by

$$\nabla_x^{\mathbf{N}_\sigma} F(\xi_\Lambda) := \sum_{k=1}^n \mathbf{1}_{\{\xi(\Lambda)=k\}} \sum_{y \in \xi_\Lambda} \mathbf{1}_{\{x=y\}} \nabla_x f_k(\xi_\Lambda), \qquad x \in \Lambda, \tag{3.2}$$

where ∇_x denotes the usual gradient on $\mathbb{R}^d$ with respect to the variable $x \in \Lambda$.

For $v \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d)$, we also let

$$\nabla_v^{\mathbf{N}_\sigma} F(\xi_\Lambda) := \sum_{y \in \xi_\Lambda} \nabla_y^{\mathbf{N}_\sigma} F(\xi_\Lambda) \cdot v(y) = \sum_{k=1}^n \mathbf{1}_{\{\xi(\Lambda)=k\}} \sum_{y \in \xi_\Lambda} \nabla_y f_k(\xi_\Lambda) \cdot v(y), \tag{3.3}$$

where $\cdot$ denotes the inner product on $\mathbb{R}^d$.

Next, we recall some results from [11]. Let $\mathrm{Diff}_0(\mathbb{X})$ be the set of all diffeomorphisms from $\mathbb{X}$ into itself with compact support, i.e., for any $\phi \in \mathrm{Diff}_0(\mathbb{X})$, there exists a compact set outside of which ϕ is the identity map. In particular, note that $\mathrm{Diff}_0(\Lambda)$ is the set of diffeomorphisms from Λ into itself. In the following, μ_ϕ denotes the image measure of μ by ϕ.

Henceforth, we assume the following technical condition.

(**H2**) : *The Radon measure μ is absolutely continuous* w.r.t. *the Lebesgue measure ℓ on $\mathbb{X}$, with Radon-Nikodym derivative $\rho = \frac{d\mu}{d\ell}$ which is strictly positive and continuously differentiable on Λ.*

Then for any $\phi \in \mathrm{Diff}_0(\Lambda)$, μ_ϕ is absolutely continuous with respect to μ with density given by

$$p_\phi^\mu(x) = \frac{d\mu_\phi(x)}{d\mu(x)} = \frac{\rho(\phi^{-1}(x))}{\rho(x)} \mathrm{Jac}(\phi^{-1})(x), \tag{3.4}$$

where $\mathrm{Jac}(\phi^{-1})(x)$ is the Jacobian of ϕ^{-1} at point $x \in \mathbb{X}$. We are now in a position to give the quasi-invariance result, see [8], [11] and [36].

Proposition 3.1 *Assume (**H1**) and (**H2**) and let η be the determinantal point process with kernel K. Then, for any measurable nonnegative f on Λ and any $\phi \in \mathrm{Diff}_0(\Lambda)$,*

$$\mathbb{E}\left[e^{-\langle f \circ \phi, \eta \rangle}\right] = \mathbb{E}\left[e^{-\langle f - \ln(p_\phi^\mu), \eta \rangle} \frac{\det J^\phi[\Lambda](\eta)}{\det J[\Lambda](\eta)}\right]. \tag{3.5}$$

We point out that the right hand side of (3.5) is well defined since $\det J[\Lambda] > 0$, $\mathbb{P}_\Lambda$-a.e.

Integration by parts

Here we give an integration by parts formula on the set of test functionals $\mathscr{S}_\Lambda$ and an extension to closed gradients and divergence operators.

We start by introducing a further condition.

($\mathbf{H3}$) : *For any $n \geq 1$, the function*

$$(x_1, \ldots, x_n) \longmapsto \det J[\Lambda](x_1, \ldots, x_n)$$

is continuously differentiable on Λ^n.

Assuming ($\mathbf{H1}$) and ($\mathbf{H3}$), we define the potential energy $U : \mathbf{N}_\sigma(\Lambda) \longrightarrow \mathbb{R}$

$$U[\Lambda](\alpha) := -\log \det J[\Lambda](\alpha)$$

and its directional derivative along $v \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d)$

$$
\begin{aligned}
\nabla_v^{\mathbf{N}_\sigma} U[\Lambda](\xi_\Lambda) \quad &:= \quad -\sum_{k=1}^\infty \mathbf{1}_{\{\xi(\Lambda)=k\}} \sum_{y \in \xi_\Lambda} \frac{\nabla_y \det J[\Lambda](\xi_\Lambda)}{\det J[\Lambda](\xi_\Lambda)} \cdot v(y) \\
&= \quad \sum_{k=1}^\infty \mathbf{1}_{\{\xi(\Lambda)=k\}} \sum_{y \in \xi_\Lambda} U_{y,k}(\xi_\Lambda) \cdot v(y). \quad\quad (3.6)
\end{aligned}
$$

The term $U_{y,k}$ in the previous definition is given by

$$U_{y,k}(\xi_\Lambda) := -\frac{\nabla_y \det J[\Lambda](\xi_\Lambda)}{\det J[\Lambda](\xi_\Lambda)} \quad \text{on } \{\xi(\Lambda) = k\}.$$

Under Condition ($\mathbf{H2}$) we define

$$\beta^\mu(x) := \frac{\nabla \rho(x)}{\rho(x)},$$

and

$$B_v^\mu(\xi_\Lambda) := \sum_{y \in \xi_\Lambda} (-\beta^\mu(y) \cdot v(y) + \operatorname{div} v(y)), \qquad v \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d),$$

where div denotes the adjoint of the gradient ∇ on Λ, i.e. div verifies

$$\int_\Lambda g(x) \operatorname{div} \nabla f(x) \, \mathrm{d}x = \int_\Lambda \nabla f(x) \cdot \nabla g(x) \, \mathrm{d}x, \quad f, g \in \mathcal{C}^\infty(\Lambda).$$

The following integration by parts formula holds, see [11].

Lemma 3.2 *Assume* ($\mathbf{H1}$), ($\mathbf{H2}$) *and* ($\mathbf{H3}$), *and let η be the determinantal point process with kernel K. Then, for any compact subset $\Lambda \subseteq \mathbb{X}$, any $F, G \in \mathscr{S}_\Lambda$ and vector field $v \in \mathcal{C}_c^\infty(\Lambda, \mathbb{R}^d)$, we have*

$$\mathbb{E}\left[G(\eta_\Lambda) \nabla_v^{\mathbf{N}_\sigma} F(\eta_\Lambda)\right] = \mathbb{E}\left[F(\eta_\Lambda) \operatorname{div}_v^{\mathbf{N}_\sigma} G(\eta_\Lambda)\right], \quad\quad (3.7)$$

where

$$\operatorname{div}_v^{\mathbf{N}_\sigma} G(\eta_\Lambda) := -\nabla_v^{\mathbf{N}_\sigma} G(\eta_\Lambda) + G(\eta_\Lambda) \left(-B_v^\mu(\eta_\Lambda) + \nabla_v^{\mathbf{N}_\sigma} U[\Lambda](\eta_\Lambda)\right).$$

Next, we extend the integration by parts formula by closability to a larger class of functionals. We refer to the Appendix for the notion of closability. Let

$$L^2_\Lambda := L^2(\mathbf{N}_\sigma(\Lambda), \mathbb{P}_\Lambda)$$

be the space of square-integrable functions with respect to $\mathbb{P}_\Lambda$. It may be checked that $\mathscr{S}_\Lambda$ is dense in L^2_Λ.

For $v \in \mathbb{C}^\infty_c(\Lambda, \mathbb{R}^d)$, we consider the linear operators $\nabla^{\mathbf{N}_\sigma}_v : \mathscr{S}_\Lambda \longrightarrow L^2_\Lambda$ and $\operatorname{div}^{\mathbf{N}_\sigma}_v : \mathscr{S}_\Lambda \longrightarrow L^2_\Lambda$ defined, respectively, by $F \mapsto \nabla^{\mathbf{N}_\sigma}_v F$ and $F \mapsto \operatorname{div}^{\mathbf{N}_\sigma}_v F$. The following theorem is proved in [11].

Theorem 3.3 *Assume* (**H1**), (**H2**), (**H3**) *and*

$$\int_{\Lambda^n} \left| \frac{\partial_{x_i^{(h)}} \det J[\Lambda](x_1, \ldots, x_n) \partial_{x_j^{(k)}} \det J[\Lambda](x_1, \ldots, x_n)}{\det J[\Lambda](x_1, \ldots, x_n)} \right|$$
$$\mathbf{1}_{\{\det J[\Lambda](x_1, \ldots, x_n) > 0\}} \, \mu(\mathrm{d}x_1) \cdots \mu(\mathrm{d}x_n) < \infty \quad (3.8)$$

for any $n \geq 1$, $1 \leq i, j \leq n$ *and* $1 \leq h, k \leq d$. *Then*

(*i*) *For any vector field* $v \in \mathbb{C}^\infty_c(\Lambda, \mathbb{R}^d)$, *the linear operators* $\nabla^{\mathbf{N}_\sigma}_v$ *and* $\operatorname{div}^{\mathbf{N}_\sigma}_v$ *are well-defined and closable. In particular, we have*

$$\nabla^{\mathbf{N}_\sigma}_v(\mathscr{S}_\Lambda) \subset L^2_\Lambda \qquad and \qquad \operatorname{div}^{\mathbf{N}_\sigma}_v(\mathscr{S}_\Lambda) \subset L^2_\Lambda.$$

(*ii*) *Denoting by* $\overline{\nabla^{\mathbf{N}_\sigma}_v}$ *(respectively* $\overline{\operatorname{div}^{\mathbf{N}_\sigma}_v}$*) the minimal closed extension of* $\nabla^{\mathbf{N}_\sigma}_v$ *(respectively* $\operatorname{div}^{\mathbf{N}_\sigma}_v$*), for any vector field* $v \in \mathbb{C}^\infty_c(\Lambda, \mathbb{R}^d)$, *we have*

$$\mathbb{E}\left[G(\eta_\Lambda) \overline{\nabla^{\mathbf{N}_\sigma}_v} F(\eta_\Lambda) \right] = \mathbb{E}\left[F(\eta_\Lambda) \overline{\operatorname{div}^{\mathbf{N}_\sigma}_v} G(\eta_\Lambda) \right],$$

for all $F \in \operatorname{Dom}\left(\overline{\nabla^{\mathbf{N}_\sigma}_v}\right)$, $G \in \operatorname{Dom}\left(\overline{\operatorname{div}^{\mathbf{N}_\sigma}_v}\right)$.

Note that under the assumptions (**H1**), (**H2**) and (**H3**), condition (3.8) is satisfied if, for any $n \geq 1$, the function

$$(x_1, \ldots, x_n) \longmapsto \det J[\Lambda](x_1, \ldots, x_n),$$

is strictly positive on the compact Λ^n.

4 Stochastic dynamics

Dirichlet forms

Assume (**H1**), and let η be the determinantal point process with kernel K. We consider the bilinear map $\mathcal{E}$ defined on $\mathscr{S}_\Lambda \times \mathscr{S}_\Lambda$ by

$$\mathcal{E}(F, G) := \mathbb{E}\left[\sum_{y \in \eta_\Lambda} \nabla^{\mathbf{N}_\sigma}_y F(\eta_\Lambda) \cdot \nabla^{\mathbf{N}_\sigma}_y G(\eta_\Lambda) \right]. \quad (4.1)$$

For $F \in \mathscr{S}_\Lambda$ of the form (3.1), i.e.

$$F(\xi_\Lambda) = f_0 \mathbf{1}_{\{\xi(\Lambda)=0\}} + \sum_{k=1}^{n} \mathbf{1}_{\{\xi(\Lambda)=k\}} f_k(\xi_\Lambda),$$

we also define the Laplacian $\mathcal{H}$ by

$$\mathcal{H}F(\xi_\Lambda) = \sum_{k=1}^{n} \mathbf{1}_{\{\xi(\Lambda)=k\}}$$
$$\sum_{y \in \xi_\Lambda} \left(-\beta^\mu(y) \cdot \nabla_y f_k(\xi_\Lambda) - \Delta_y f_k(\xi_\Lambda) + U_{y,k}(\xi_\Lambda) \cdot \nabla_y f_k(\xi_\Lambda) \right),$$

where $\Delta = -\mathrm{div}\nabla$ denotes the Laplacian operator on $\mathbb{R}^d$.

In the following, we consider the subspace $\tilde{\mathscr{S}}_\Lambda$ of $\mathscr{S}_\Lambda$ consisting of function $F \in \mathscr{S}_\Lambda$ of the form

$$F(\xi_\Lambda) = f\left(\langle \phi_1, \xi_\Lambda \rangle, \dots, \langle \phi_M, \xi_\Lambda \rangle \right) \mathbf{1}_{\{\xi(\Lambda) \leq K\}},$$

for some integers $M, K \geq 1$, $\phi_1, \dots, \phi_M \in \mathcal{C}^\infty(\Lambda)$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^M)$. Note that $\tilde{\mathscr{S}}_\Lambda$ is dense in L_Λ^2 (see e.g. [25] p. 54).

Theorem 4.1 below is proved in [11]. We refer the reader to the Appendix for the required notions of Dirichlet forms theory.

Theorem 4.1 *Under the assumptions of Theorem 3.3, we have*
(i) The linear operator $\mathcal{H} : \tilde{\mathscr{S}}_\Lambda \longrightarrow L_\Lambda^2$ is symmetric, non-negative definite and well-defined, i.e. $\mathcal{H}(\tilde{\mathscr{S}}_\Lambda) \subset L_\Lambda^2$. In particular the operator square root $\mathcal{H}^{1/2}$ of $\mathcal{H}$ exists.
(ii) The bilinear form $\mathcal{E} : \tilde{\mathscr{S}}_\Lambda \times \tilde{\mathscr{S}}_\Lambda \longrightarrow \mathbb{R}$ is symmetric, non-negative definite and well-defined, i.e. $\mathcal{E}(\tilde{\mathscr{S}}_\Lambda \times \tilde{\mathscr{S}}_\Lambda) \subset \mathbb{R}$.
(iii) $\mathcal{H}^{1/2}$ and $\mathcal{E}$ are closable and the following relation holds:

$$\overline{\mathcal{E}}(F, G) = \mathbb{E}[\overline{\mathcal{H}^{1/2}} F(\eta_\Lambda) \overline{\mathcal{H}^{1/2}} G(\eta_\Lambda)], \quad \forall \, F, G \in \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}). \qquad (4.2)$$

(iv) The bilinear form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$ is a symmetric Dirichlet form.

Associated diffusion processes

We start recalling some notions, see Chapters IV and V in [25]. We call $\mathbf{N}$ the space of $\mathbb{N}$-valued Radon measures on $\mathbb{X}$, as opposed to $\mathbf{N}_\sigma$ the space of *simple* $\mathbb{N}$-valued Radon measures on $\mathbb{X}$. We denote by $\mathbf{N}(\Lambda)$ the space of $\mathbb{N}$-valued Radon measures supported on a compact $\Lambda \subseteq \mathbb{X}$. We equip $\mathbf{N}$ with the vague topology, and denote by $\mathscr{N}$ the corresponding Borel σ-algebra and by $\mathscr{N}(\Lambda)$ the corresponding trace-σ-algebra. Given π in the set $\mathrm{P}(\mathbf{N}(\Lambda))$ of the probability measures on $(\mathbf{N}(\Lambda), \mathscr{N}(\Lambda))$, we call a π-stochastic process with state space $\mathbf{N}(\Lambda)$ the collection

$$\mathbf{M}_{\Lambda,\pi} = (\mathbf{\Omega}, \mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, (\mathbf{M}_t)_{t \geq 0}, (\mathbf{P}_\xi)_{\xi \in \mathbf{N}(\Lambda)}, \mathbf{P}_\pi),$$

where $\mathcal{A} := \bigvee_{t \geq 0} \mathcal{A}_t$ is a σ-algebra on the set $\boldsymbol{\Omega}$, $(\mathcal{A}_t)_{t \geq 0}$ is the $\mathbf{P}_\pi$-completed filtration generated by the process $\mathbf{M}_t : \boldsymbol{\Omega} \longrightarrow \mathbf{N}(\Lambda)$, $\mathbf{P}_\xi$ is a probability measure on $(\boldsymbol{\Omega}, \mathcal{A})$ for all $\xi \in \mathbf{N}(\Lambda)$, and $\mathbf{P}_\pi$ is the probability measure on $(\boldsymbol{\Omega}, \mathcal{A})$ defined by

$$\mathbf{P}_\pi(A) := \int_{\mathbf{N}(\Lambda)} \mathbf{P}_\xi(A)\, \pi(\mathrm{d}\xi), \quad A \in \mathcal{A}.$$

A collection $(\mathbf{M}_{\Lambda,\pi}, (\theta_t)_{t \geq 0})$ is called a π-time homogeneous Markov process with state space $\mathbf{N}(\Lambda)$ if $\theta_t : \boldsymbol{\Omega} \longrightarrow \boldsymbol{\Omega}$ is a shift operator, i.e. $\mathbf{M}_s \circ \theta_t = \mathbf{M}_{s+t}$, $s, t \geq 0$, the map $\xi \mapsto \mathbf{P}_\xi(A)$ is measurable for all $A \in \mathcal{A}$, and the time homogeneous Markov property

$$\mathbf{P}_\xi(\mathbf{M}_t \in A \,|\, \mathcal{A}_s) = \mathbf{P}_{\mathbf{M}_s}(\mathbf{M}_{t-s} \in A), \ \mathbf{P}_\xi - a.s., \ A \in \mathcal{A}, \ 0 \leq s \leq t, \ \xi \in \mathbf{N}(\Lambda),$$

holds. Recall that a π-time homogeneous Markov process $(\mathbf{M}_{\Lambda,\pi}, (\theta_t)_{t \geq 0})$ with state space $\mathbf{N}(\Lambda)$ is said to be π-tight on $\mathbf{N}(\Lambda)$ if $(\mathbf{M}_t)_{t \geq 0}$ is right-continuous with left limits $\mathbf{P}_\pi$-almost surely; $\mathbf{P}_\xi(\mathbf{M}_0 = \xi) = 1 \ \forall \xi \in \mathbf{N}(\Lambda)$; the filtration $(\mathcal{A}_t)_{t \geq 0}$ is right continuous; the following strong Markov property holds:

$$\mathbf{P}_{\pi'}(\mathbf{M}_{t+\tau} \in A \,|\, \mathcal{A}_\tau) = \mathbf{P}_{\mathbf{M}_\tau}(\mathbf{M}_t \in A)$$

$\mathbf{P}_{\pi'}$-almost surely for all $\mathcal{A}_t$-stopping time τ, $\pi' \in \mathrm{P}(\mathbf{N}(\Lambda))$, $A \in \mathcal{A}$ and $t \geq 0$, cfr. Theorem IV.1.15 in [25]. In addition, a π-tight process on $\mathbf{N}(\Lambda)$ is said to be a π-special standard process on $\mathbf{N}(\Lambda)$ if for any $\pi' \in \mathrm{P}(\mathbf{N}(\Lambda))$ which is equivalent to π and all $\mathcal{A}_t$-stopping times τ, $(\tau_n)_{n \geq 1}$ such that $\tau_n \uparrow \tau$ we have that $\mathbf{M}_{\tau_n}$ converges to $\mathbf{M}_\tau$, $\mathbf{P}_{\pi'}$-almost surely.

The following theorem is proved in [11]. Therein $\mathbf{E}_\xi$ denotes the expectation under $\mathbf{P}_\xi$, $\xi \in \mathbf{N}(\Lambda)$. Here again, we refer the reader to the Appendix for the required notions of Dirichlet forms theory.

Theorem 4.2 *Assume the hypotheses of Theorem 3.3, let $\mathbb{P}$ be the law of a determinantal point process η with kernel K, and $\overline{\mathcal{E}}$ be the Dirichlet form constructed in Theorem 4.1. Then there exists a $\mathbb{P}_\Lambda$-tight special standard process $(\mathbf{M}_{\Lambda, \mathbb{P}_\Lambda}, (\theta_t)_{t \geq 0})$ on $\mathbf{N}(\Lambda)$ such that:*

1. *$\mathbf{M}_{\Lambda, \mathbb{P}_\Lambda}$ is a diffusion, in the sense that:*

$$\mathbf{P}_\xi(\{\omega : \ t \mapsto \mathbf{M}_t(\omega) \text{ is continuous on } [0, +\infty)\}) = 1, \qquad \overline{\mathcal{E}}\text{-}a.e. \ \xi \in \mathbf{N}(\Lambda); \tag{4.3}$$

2. *the transition semigroup of $\mathbf{M}_{\Lambda, \mathbb{P}_\Lambda}$ is given by*

$$p_t F(\xi) := \mathbf{E}_\xi[F(\mathbf{M}_t)], \quad \xi \in \mathbf{N}(\Lambda), \quad F : \mathbf{N}(\Lambda) \longrightarrow \mathbb{R} \quad \text{square integrable,}$$

and it is properly associated with the Dirichlet form $(\overline{\mathcal{E}}, \mathrm{Dom}(\overline{\mathcal{H}^{1/2}}))$, i.e. $p_t F$ is an $\overline{\mathcal{E}}$-a.c., $\mathbb{P}_\Lambda$-version of $\exp(-t\mathcal{H}_\Lambda^{gen})F$, for all square integrable $F : \mathbf{N}(\Lambda) \longrightarrow \mathbb{R}$ and $t > 0$ (where $\mathcal{H}_\Lambda^{gen}$ is the generator of $\overline{\mathcal{E}}$);

3. *$\mathbf{M}_{\Lambda, \mathbb{P}_\Lambda}$ is unique up to $\mathbb{P}_\Lambda$-equivalence (we refer the reader to Definition 6.3 page 140 in [26] for the meaning of this notion);*

4. $\mathbf{M}_{\Lambda,\mathbb{P}_\Lambda}$ *is* $\mathbb{P}_\Lambda$*-symmetric, i.e.*

$$\mathbb{E}\left[G(\eta_\Lambda)\, p_t F(\eta_\Lambda)\right] = \mathbb{E}\left[F(\eta_\Lambda)\, p_t G(\eta_\Lambda)\right],$$

for square integrable functions F *and* G *on* $\mathbf{N}(\Lambda)$*;*

5. $\mathbf{M}_{\Lambda,\mathbb{P}_\Lambda}$ *has* $\mathbb{P}_\Lambda$ *as invariant measure.*

In dimension $d \geq 2$, the diffusion constructed in the previous theorem is non-colliding. Indeed, the following theorem holds.

Theorem 4.3 *Assume* $d \geq 2$*, and the hypotheses of Theorem 3.3. Then*

$$\mathbf{P}_\xi(\{\omega \in \mathbf{\Omega} : \mathbf{M}_t(\omega) \in \mathbf{N}_\sigma(\Lambda), \quad for\ any\ t \in [0,\infty)\}) = 1, \qquad \overline{\mathcal{E}}\text{-}a.e.\ \xi \in \mathbf{N}_\sigma(\Lambda).$$

An illustrating example

Let $\Lambda := B(0, R) \subset \mathbb{R}^2$ be the closed ball centered at the origin with radius $R \in (0, 1)$, let $\{\varphi_k^{(R)}\}_{1 \leq k \leq 3}$, denote the orthonormal subset of $L^2(B(0, R), \ell)$ defined by

$$\varphi_k^{(R)}(x) := \frac{1}{R}\sqrt{\frac{k+1}{\pi}}\left(\frac{x^{(1)}}{R} + i\frac{x^{(2)}}{R}\right)^k, \quad x = (x^{(1)}, x^{(2)}) \in B(0, R),\ k = 1, 2, 3,$$

where $\mu = \ell$ is the Lebesgue measure on $\mathbb{R}^2$ and $i := \sqrt{-1}$ denotes the complex unit. We consider the truncated Bergman kernel (see [18]) restricted to Λ

$$K_{\mathrm{Be}}(x, y) := \sum_{k=1}^3 R^{2(k+1)}\varphi_k^{(R)}(x)\overline{\varphi_k^{(R)}(y)}, \qquad x, y \in B(0, R),$$

and denote by $\mathcal{K}_{\mathrm{Be}}$ the associated integral operator.

We now discuss the conditions of Theorem 4.2. First, K_{Be} is readily seen to be Hermitian and locally of trace class with non-zero eigenvalues $\kappa_k := R^{2(k+1)}$, $k = 1, 2, 3$. As a consequence, the spectrum of $\mathcal{K}_{\mathrm{Be}}$ is contained in $[0, 1)$ and the triplet $(\mathcal{K}_{\mathrm{Be}}, K_{\mathrm{Be}}, \ell)$ satisfies assumption (**H1**). In addition, Condition (**H2**) is trivially satisfied since $\mu = \ell$ is the Lebesgue measure.

Denoting by η_Λ the determinantal point process with kernel K_{Be}, the Janossy densities of η_Λ are given by

$$j_\Lambda^{(k)}(x_1, \ldots, x_k) = \mathrm{Det}\,(\mathrm{Id} - \mathcal{K}_{\mathrm{Be}})\det J[\Lambda](x_1, \ldots, x_k),$$

for $k = 1, 2, 3$, $(x_1, \ldots, x_k) \in \Lambda^k$, and where the kernel $J[\Lambda]$ of $\mathcal{J}[\Lambda]$ is given by

$$J[\Lambda](x, y) := \sum_{h=1}^3 \frac{R^{2(h+1)}}{1 - R^{2(h+1)}}\varphi_h^{(R)}(x)\overline{\varphi_h^{(R)}(y)}.$$

Moreover, η_Λ has at most 3 points according to Proposition 2.5, which means that $j_\Lambda^k = 0$, for $k \geq 4$. To prove condition (**H3**) it suffices to remark that the function

$$(x_1, \ldots, x_k) \to \det(J[\Lambda](x_p, x_q))_{1 \leq p,q \leq k}$$

is continuously differentiable on Λ^k, for $k \le 3$. Condition (3.8) is trivially satisfied for $k > 3$ since as already observed in this case $j_\Lambda^k = 0$. Next, we check that Condition (3.8) is verified for $k = 3$. To that end, note that

$$J[\Lambda](x_1, x_2, x_3) = A(x_1, x_2, x_3)A(x_1, x_2, x_3)^*,$$

where the matrix $A := (A_{ph})_{1 \le p, h \le 3}$ is given by

$$A_{ph} := \frac{R^{h+1}}{\sqrt{1 - R^{2(h+1)}}} \varphi_h^{(R)}(x_p)$$

and $A(x_1, x_2, x_3)^*$ denotes the transpose conjugate of $A(x_1, x_2, x_3)$. Hence,

$$\det J[\Lambda](x_1, x_2, x_3) = |\det A(x_1, x_2, x_3)|^2,$$

and since the previous determinant is a Vandermonde determinant, we have

$$\det A(x_1, x_2, x_3) = \prod_{p=1}^{3} \sqrt{\frac{1+p}{\pi(1 - R^{2(p+1)})}} \left(\prod_{p=1}^{3} (x_p^{(1)} + i x_p^{(2)}) \right)$$
$$\prod_{1 \le p < q \le 3} ((x_p^{(1)} - x_q^{(1)}) + i(x_p^{(2)} - x_q^{(2)})).$$

So, Condition (3.8) with $k = 3$ reduces to

$$\int_{B(0,R)^3} \left| \frac{\partial_{x_i^{(h)}} |\det A(x_1, x_2, x_3)|^2 \partial_{x_j^{(k)}} |\det A(x_1, x_2, x_3)|^2}{|\det A(x_1, x_2, x_3)|^2} \right| \ell(\mathrm{d}x_1)\ell(\mathrm{d}x_2)\ell(\mathrm{d}x_3) < \infty,$$

for all $1 \le i, j \le 3$ and $1 \le h, k \le 2$, and for this it suffices to check

$$\int_{B(0,R)^3} \left| \frac{\partial_{x_1^{(1)}} |\det A(x_1, x_2, x_3)|^2}{|\det A(x_1, x_2, x_3)|^2} \right| \ell(\mathrm{d}x_1)\ell(\mathrm{d}x_2)\ell(\mathrm{d}x_3) < \infty.$$

This latter integral can be written as

$$\int_{B(0,R)^3} \left| \frac{2x_1^{(1)}}{(x_1^{(1)})^2 + (x_1^{(2)})^2} + 2\sum_{j=2}^{3} \frac{x_1^{(1)} - x_j^{(1)}}{(x_1^{(1)} - x_j^{(1)})^2 + (x_1^{(2)} - x_j^{(2)})^2} \right| \ell(\mathrm{d}x_1)\ell(\mathrm{d}x_2)\ell(\mathrm{d}x_3),$$

which is indeed finite. Condition (3.8) may be verified also for $k < 3$ by taking into account some properties of generalized Vandermonde determinants, we refer the reader to [11] for the details. Consequently, by Theorem 4.2 we have the existence of a diffusion process properly associated to the determinantal point process with the Bergman-type kernel K_{Be}.

5   Simulation

Standard simulation of determinantal point processes

In this section, we describe the standard algorithm to sample from the law of a determinantal point process. The main results of this section can be found in

the seminal work of [17], along with the improvements found in [12], [18], [22]. We recall the algorithm introduced there in order to insist on its advantages and disadvantages compared to directly simulating according to the densities. The standard algorithm first yields a way to simulate the number of points $n \in \mathbb{N}$ of a determinantal point process on a given compact $\Lambda \subseteq \mathbb{X}$. Second, it provides a sample from the Janossy density j_Λ^n. Let us now discuss in detail these two steps.

Theorem 5.1 *Let $\mathcal{K}$ be a trace class integral operator satisfying* **(H1)** *(we often take $\mathcal{K}_\Lambda$, which is indeed of trace class), $\{\varphi_n\}_{n \geq 1}$ an orthonormal basis of $L^2(\mathbb{X}, \mu)$ formed by eigenfunctions of $\mathcal{K}$ and $\{\mu_n\}_{n \geq 1}$ the corresponding sequence of eigenvalues. We write*

$$K(x, y) = \sum_{n \geq 1} \mu_n \varphi_n(x) \overline{\varphi_n(y)}, \quad x, y \in \mathbb{X}. \tag{5.1}$$

Let $\{B_n\}_{n \geq 1}$ be a sequence of independent Bernoulli random variables of mean $\mathbb{E}[B_n] = \mu_n$. The Bernoulli random variables are defined on a distinct probability space, say $(\Omega, \mathcal{F})$. Then, define the (random) kernel

$$K_B(x, y) = \sum_{n \geq 1} B_n \varphi_n(x) \overline{\varphi_n(y)}, \quad x, y \in \mathbb{X}.$$

Finally, define the point process η on $(\mathbf{N}_\sigma \times \Omega, \mathcal{N}_\sigma \otimes \mathcal{F})$ as the point process obtained by first drawing the Bernoulli random variables, and then the point process with kernel K_B. We have that η is a determinantal point process on $\mathbb{X}$ with kernel K.

For the remainder of this paragraph, we consider a general kernel K of the form (5.1) and wish to generate a sample of the determinantal point process with kernel $\mathcal{K}$.

According to Theorem 5.1, the number of points on $\mathbb{X}$ is distributed as the sum of independent Bernoulli random variables. More precisely,

$$|\xi(\mathbb{X})| \sim \sum_{n \geq 1} B_n,$$

where $B_n \sim \text{Be}(\mu_n)$, $n \in \mathbb{N}$. Define $T := \sup\{n \geq 1 \ / \ B_n = 1\}$. Since $\sum_{n \geq 1} \mu_n = \sum_{n \geq 1} \mathbb{P}(B_n = 1) < \infty$, by a direct application of the Borel-Cantelli lemma, we have that $T < \infty$ almost surely. Hence the method is to simulate first a realization of T, say t, and then $t - 1$ independent Bernoulli random variables $B_1, \ldots, B_{t-1}$, each B_n with mean μ_n, $n = 1, \ldots, t - 1$ Finally, set $B_t = 1$.

The simulation of the random variable T can be obtained by the inversion method, as we know its cumulative distribution function explicitly. Indeed, for $n \in \mathbb{N}$,

$$\mathbb{P}(T = n) = \mu_n \prod_{i=n+1}^{\infty} (1 - \mu_i),$$

hence

$$F(r) = \mathbb{P}(T \le r) = \sum_{n \le r} \mu_n \prod_{i=n+1}^{\infty} (1 - \mu_i), \quad \forall r \in \mathbb{N}. \qquad (5.2)$$

To generate a random variable with law F requires the numerical computation of the generalized inverse $F^{-1}(u) := \inf\{t \in \mathbb{N} \,/\, F(t) \ge u\}$. In many practical cases, as in the case of the Ginibre point process, the numerical calculations may augment the complexity of the algorithm and the CPU. This is the main reason for which we shall propose an approximate simulation of the Ginibre point process.

Assume we have simulated the number of points of the determinantal point process on a compact Λ. For the clarity, we suppose $T = n$ and $B_1 = 1, B_2 = 1, \ldots, B_n = 1$. This assumption is equivalent to a simple reordering of the eigenvectors $(\varphi_n)_{n \in \mathbb{N}}$. Then we have reduced the problem to that of simulating the vector $(X_1, \ldots, X_n)$ of joint density

$$p(x_1, \ldots, x_n) = \frac{1}{n!} \det\left(\tilde{K}(x_i, x_j)\right)_{1 \le i,j \le n},$$

where $\tilde{K}(x, y) = \sum_{j=1}^{n} \psi_j(x)\overline{\psi_j(y)}$, for $x, y \in \Lambda$, where here $(\psi_j)_{j \in \mathbb{N}}$ is the reordering of $(\varphi_j)_{j \in \mathbb{N}}$. The determinantal point process of kernel $\tilde{K}$ has n points almost surely by Proposition 2.5, which means that it remains to simulate the unordered vector $(X_1, \ldots, X_n)$ of points of the point process. The idea of the algorithm is to start by simulating X_1, then $X_2|X_1$, until $X_n|X_1, \ldots, X_{n-1}$. The key here is that in the determinantal case, the density of these conditional probabilities takes a computable form. Let us start by observing that

$$\det\left(\tilde{K}(x_i, x_j)\right)_{1 \le i,j \le n_i} = \det\left(\psi_k(x_l)\right)_{1 \le k,l \le n_i} \det\left(\overline{\psi_l(x_k)}\right)_{1 \le k,l \le n},$$

so the density of X_1 on Λ is

$$p_1(x_1) = \int \ldots \int p(x_1, \ldots, x_n)\, \mu(\mathrm{d}x_2) \cdots \mu(\mathrm{d}x_n)$$

$$= \frac{1}{n!} \sum_{\tau, \sigma \in S_n} \mathrm{sgn}(\tau)\mathrm{sgn}(\sigma)\psi_{\tau(1)}(x_1)\overline{\psi_{\sigma(1)}(x_1)} \prod_{k=2}^{n} \int \psi_{\tau(k)}(x_k)\overline{\psi_{\sigma(k)}(x_k)}\, \mu(\mathrm{d}x_k)$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} |\psi_{\sigma(1)}(x_1)|^2$$

$$= \frac{1}{n} \sum_{k=1}^{n} |\psi_k(x_1)|^2,$$

where S_n is the n-th symmetric group and $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_n$. By a similar computation, we may compute the distribution of $X_2|X_1$, whose density with respect to μ is given by

$$p_{X_2|X_1}(x_2) = \frac{p_2(X_1, x_2)}{p_1(X_1)} = \frac{1}{(n-1)! \sum |\psi_j(X_1)|^2}$$

$$\sum_{\sigma \in S_n} \left(|\psi_{\sigma(1)}(X_1)|^2|\psi_{\sigma(2)}(x_2)|^2 - \psi_{\sigma(1)}(X_1)\overline{\psi_{\sigma(2)}}(X_1)\psi_{\sigma(2)}(x_2)\overline{\psi_{\sigma(1)}}(x_2)\right)$$

$$= \frac{1}{n-1} \left(\sum_{i=1}^{n} |\psi_j(x_2)|^2 - |\sum_{j=1}^{n} \frac{\psi_j(X_1)}{\sqrt{\sum |\psi_j(X_1)|^2}} \overline{\psi_j}(x_2)|^2 \right).$$

The previous formula can be generalized recursively, and has the advantage of giving a natural interpretation of the conditional densities. Indeed, we may write the conditional densities at each step in a way that makes the orthogonalization procedure appear. This is presented in the final algorithm, which was explicitly obtained in [22] (see also [17] for the proof). We define the vector $\mathbf{v}(x) := (\psi_1(x), \ldots, \psi_n(x))^t$, where t stands for the transpose operator, denote by $\|\mathbf{v}(x)\|$ its Euclidean norm, and given $\mathbf{x} \in \mathbb{C}^n$, we set $\mathbf{x}^* := \overline{\mathbf{x}}^t$.

Algorithm 1 Simulation of the determinantal projection point process

    sample X_n from the distribution with density $p_n(x) = \|\mathbf{v}(x)\|^2/n$, $x \in \Lambda$
    $\mathbf{e}_1 \leftarrow \mathbf{v}(X_n)/\|\mathbf{v}(X_n)\|$
    for $j = n-1 \rightarrow 1$ **do**
      sample X_j from the distribution with density

$$p_j(x) = \frac{1}{j} \left[\|\mathbf{v}(x)\|^2 - \sum_{k=1}^{n-j} |\mathbf{e}_k^* \mathbf{v}(x)|^2 \right]$$

      $\mathbf{w}_j \leftarrow \mathbf{v}(X_j) - \sum_{k=1}^{n-j} (\mathbf{e}_k^* \mathbf{v}(X_j)) \mathbf{e}_k, \quad \mathbf{e}_{n-j+1} \leftarrow \mathbf{w}_j/\|\mathbf{w}_j\|$
    end for
    return $(X_1, \ldots, X_n)$

It is then known that Algorithm 1 yields a sample $\{X_1, \ldots, X_n\}$ of a determinantal point process with kernel $\tilde{K}(x, y) = \sum_{j=1}^{n} \psi_j(x)\overline{\psi_j(y)}$, $x, y \in \Lambda$.

Simulation using Markov chains

Exploiting the bound (2.15), an alternative algorithm to sample from the law of a determinantal point process on a finite window is readily obtained by specializing the general theory developed in [19], [20] and [23], which allow to sample from the law of finite point processes with bounded Papangelou intensity. Let us give a brief description.

In the remainder of this paragraph, we fix a compact set $\Lambda \subseteq \mathbb{X}$, and turn our attention to the simulation of a determinantal point process with kernel K_Λ. The following bound holds for the Papangelou conditional intensity c_Λ:

$$\forall x \in \Lambda, \ \forall \xi \in \mathbf{N}_\sigma, \ c_\Lambda(x, \xi) \leq J[\Lambda](x, x) = J(x, x), \tag{5.3}$$

where we have specialized the bound (2.15), and have noticed that $J[\Lambda](x, x) = J(x, x)$ for $x \in \Lambda$. We first simulate a Glauber process associated to the measure $J(x, x)\mathrm{d}\mu(x)$:

- Draw an initial configuration D_0 according to the distribution of a Poisson point process over Λ with mean measure $J(x, x)\mathrm{d}\mu(x)$.

- Define a Poisson process on $\mathbb{R}_+$ of intensity $M = \int_\Lambda J(x,x)\mathrm{d}\mu(x)$ and denote by $(T_n, n \geq 1)$ its arrival times.

- At each time T_n, a particle appears at a position randomly located according to the probability distribution $M^{-1}J(x,x)\mathrm{d}\mu(x)$ independently from any other event.

- To each particle, we assign an exponentially distributed lifetime of mean 1, independently from any other event, i.e. each particle dies after an exponential distributed time.

- The Glauber process D is formed by the random variables D_t denoting the number of particles alive at time t.

Once this process is constructed, we can use the coupling from the past to simulate the determinantal point process:

- Simulate a (dominating) Glauber process D corresponding to the mean measure $J(x,x)\mathrm{d}\mu(x)$ over Λ on a time horizon T, with initial configuration D_0. Record all birth dates and locations along the sample-path.

- Define two configuration-valued Markov chains, L and U. L stands for *lower* and U for *upper* since we will guarantee $L_t \subset U_t \subset D_t$ at any time $t \geq 0$, $L_0 = \emptyset$ and $U_0 = D_0$.

- Read the time-line of the process D.

    1. When there is a death in the sample-path of D, then the corresponding particle dies (in both U and L) provided it exists.

    2. When there is a birth at x in D at time t, draw a uniform sample S on $[0,1]$, independently from everything else. If $S \leq c(x, U_{t^-})$ then x is added to $L_t = L_{t^-} \cup x$. If $S \leq c(x, L_{t^-})$, then $U_t = U_{t^-} \cup \{x\}$.

    3. If at time T, $U_T = L_T$ then U_T is a sample of the determinantal point process of Papangelou intensity c. If not, expand the sample-path of D to $[T, 2T]$ and replay the same algorithm.

A crucial question is then how to choose T to avoid both a too long simulation if T is large and the need to extend several times the sample-path of D if T is too small. A very crude bound on the coalescence time, i.e. the time at which U and L coincide, is the hitting time of the null configuration by D. Indeed, since for any time t, $L_t \subset U_t \subset D_t$, if $D_T = \emptyset$ then $U_T = \emptyset$ and $L_T = \emptyset$. It turns out that the number of points of D follow the dynamics of an M/M/∞ queue. If the initial population of D_0 is large then Proposition 6.8 of [30] entails that T_0 is of the order of $\log(|D_0|)$. This means that the coalescence time of our algorithm is an $O\left(\log \int_\Lambda J(x,x)\,\mathrm{d}\mu(x)\right)$, but in practice, we are well below this upper bound.

Finally, we present some samples of the coalescence time in a practical example known as the Gaussian model (see [22]). More precisely, Figure 2(a) shows the distributions of the coalescence time of L_t and U_t and the stopping time of the algorithm for 500 samples of the Gaussian model DPP with $\rho = 50$.

The two simulation methods presented are conceptually quite different and are therefore difficult to compare. To be more precise, in the standard algorithm, there are two time-consuming steps: the simulation of the Bernoulli
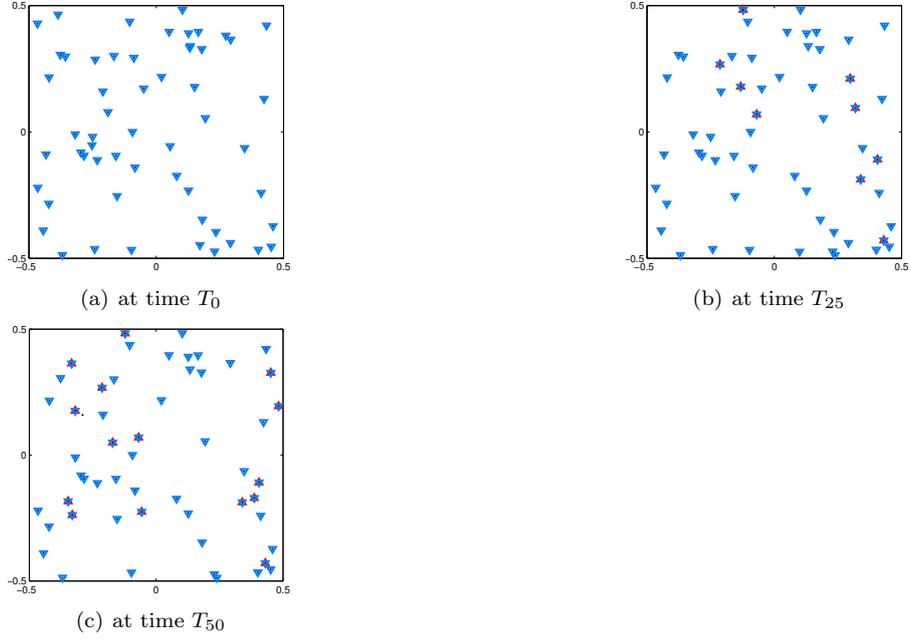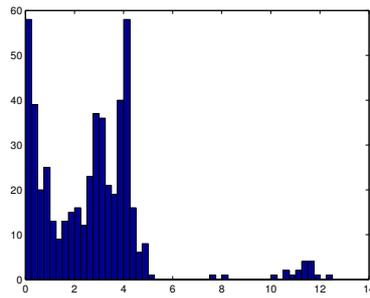
(a) at time T_0

(b) at time T_{25}

(c) at time T_{50}

Figure 1: CFTP simulations for Gaussian model DPP with $\rho = 50$ and $\alpha = 0.04$, respectively at time T_i, the i-th jump time from time $t = -n$. Notations: "$\cdot$" := D_t, "∇" := U_t and "Δ"(red) := L_t.

(a) Coalescence time of L_t and U_t

Figure 2: Histogram of the coalescence time of L_t and U_t and the stopping time on 500 samples of a Gaussian model with $\rho = 50$ and $\alpha = 0.04$.

random variables and the simulation under the density p_i for which we are a priori required to proceed by rejection sampling. This requires an evaluation of the supremum of p_i on a grid which can be unboundedly big. In the algorithm based on Markov chains, we avoid the previous problem by only evaluating elaborate functionals (in our case, the Papangelou conditional intensity c) on a specific configuration, and not on the whole grid. Additionally, the standard algorithm relies on the knowledge of the eigenfunctions and the eigenvalues of the kernel K_Λ whereas the algorithm based on Markov chains works well with any expression of $J[\Lambda]$. However, the time necessary to reach equilibrium can be quite long, which is the main drawback of this algorithm. Thus, quantifying the execution time of the MCMC algorithm is of practical interest. We roughly discussed this question in this section, but a comparison with the standard algorithm is in general quantitatively difficult since the better performing algorithm depends on the kernel K_Λ of the underlying determinantal point process.

Approximate simulation of the Ginibre point process

In this paragraph, we introduce a specific determinantal point process which is fast to simulate in practice, well-suited for applications, and converges weakly to the Ginibre point process.

The Ginibre point process, see [15], is the determinantal point process on $\mathbb{C}$ with kernel

$$K_{\mathrm{Gin}}(z_1, z_2) := \sum_{n=0}^{\infty} \phi_n(z_1)\overline{\phi_n(z_2)}, \quad z_1, z_2 \in \mathbb{C}, \qquad (5.4)$$

where $\phi_n(z) := \frac{1}{\sqrt{\pi n!}} e^{-\frac{1}{2}|z|^2} z^n$ for each $n \geq 0$. Further details concerning the Ginibre point process may be found in [18] and [31].

We introduce a new kernel, by setting

$$K_{\mathrm{Gin}}^N(z_1, z_2) := \sum_{n=0}^{N-1} \phi_n^{\sqrt{N}}(z_1)\overline{\phi_n^{\sqrt{N}}(z_2)}, \quad z_1, z_2 \in B(0, \sqrt{N}), \qquad (5.5)$$

where we define $\phi_n^{\sqrt{N}} := \frac{1}{\sqrt{\pi\gamma(n+1,N)}} e^{-\frac{1}{2}|z|^2} z^n \mathbf{1}_{\{z \in B(0,\sqrt{N})\}}$, for $0 \leq n \leq N-1$. Here, $\gamma(z,a) := \int_0^a e^{-t} t^{z-1}\, \mathrm{d}t$, $a \geq 0, z \in \mathbb{C}$ is the lower incomplete Gamma function. This kernel defines a determinantal point process named truncated Ginibre point process conditioned on having N points, see [12] for details. Clearly, this determinantal point process can be simulated as described by Algorithm 1. The fact that we know the number of points N in the ball $B(0, \sqrt{N})$ trivially ensures a fast execution time.

As already noticed, Algorithm 1 yields a sample of the truncated Ginibre point process conditioned on having N points on the ball $B(0, \sqrt{N})$. In order to simulate the process on $B(0, a)$, $a \geq 0$, we need to apply a homothetic transformation to the N points, which translates to a homothety on the eigenfunctions. To summarize, the simulation algorithm for the truncated Ginibre process conditioned on having N points on the ball $B(0, a)$ is done according to Algorithm 2.

Algorithm 2 Simulation of the truncated Ginibre point process

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define $\phi_k(z) = \frac{N}{\pi a^2 \gamma(k+1,N)} e^{-\frac{N}{2a^2}|z|^2} (\frac{Nz}{a^2})^k$, for $z \in B(0, \sqrt{N})$ and $0 \leq k \leq N-1$.

define $\mathbf{v}(z) := (\phi_0(z), \ldots, \phi_{N-1}(z))$, for $z \in B(0, \sqrt{N})$.

sample X_N from the distribution with density $p_N(z) = \|\mathbf{v}(z)\|^2/N$, $z \in B(0, \sqrt{N})$

set $\mathbf{e}_1 = \mathbf{v}(X_N)/\|\mathbf{v}(X_N)\|$

for $i = N-1 \to 1$ **do**

    sample X_i from the distribution with density

$$p_i(x) = \frac{1}{i}\left[\|\mathbf{v}(x)\|^2 - \sum_{j=1}^{N-i} |\mathbf{e}_j^* \mathbf{v}(x)|^2 \right]$$

    set $\mathbf{w}_i = \mathbf{v}(X_i) - \sum_{j=1}^{N-i} \left(\mathbf{e}_j^* \mathbf{v}(X_i)\right) \mathbf{e}_j$, $\quad \mathbf{e}_{N-i+1} = \mathbf{w}_i/\|\mathbf{w}_i\|$

end for

return $(X_1, \ldots, X_N)$

The next theorem from [12] and the subsequent comment guarantee that the above algorithm can be interpreted as an approximate simulation algorithm for the Ginibre point process.

Theorem 5.2 *The kernel K_{Gin}^N converges to K_{Gin}, as N tends to infinity, uniformly on compacts.*

As a consequence of Theorem 5.2 and Proposition 3.10 in [32], the truncated Ginibre point process conditioned on having N points converges weakly to the Ginibre point process.

6   Open questions

We mention here a few open questions.

- Let $\mathbb{P}$ be the law of a determinantal point process η on $\mathbb{X}$, and ϕ a diffeomorphism of the whole space. Is the image of $\mathbb{P}$ by ϕ absolutely continuous with respect to $\mathbb{P}$? If yes, is it possible to compute the corresponding Radon-Nikodym derivative?

- Is the diffusion constructed in Theorem 4.2 ergodic?

- Consider a sequence of diffusions defined by Theorem 4.2 and indexed by compacts Λ_n increasing to $\mathbb{R}^d$. Does $\mathbf{M}_{\Lambda_n, \mathbb{P}_{\Lambda_n}}$ converge weakly to some limiting diffusion as $n \to \infty$? If yes, may we compute the properly associated Dirichlet form?

- Is it possible to approximate in distribution the diffusion constructed in Theorem 4.2 by a continuous-time Markov process (such as a Glauber dynamics)?

- What is the error committed by the approximate simulation algorithm to sample from the target law, i.e. the law of the Ginibre point process?

- Let η be a determinantal point process with integral operator $\mathcal{K}$. Can one generalize the results presented in this chapter to include the case where 1 is an eigenvalue of $\mathcal{K}$?

7 Appendix

First, we recall some results and properties on the closability of linear operators. Given $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ two Banach spaces, and $A : \mathrm{Dom}(A) \longrightarrow Y$ a linear operator defined on a subspace $\mathrm{Dom}(A)$ of X, the domain of A, the operator A is said to be closed if, for any sequence $(x_n)_{n \geq 1} \subset \mathrm{Dom}(A)$, such that x_n converges to x in X and Ax_n converges to y in Y we have $x \in \mathrm{Dom}(A)$ and $y = Ax$, i.e. $\mathrm{Dom}(A)$ is closed (or equivalently complete) w.r.t. the graph norm $\|\cdot\|_G := \|\cdot\|_X + \|A\cdot\|_Y$. A linear operator $A : \mathrm{Dom}(A) \longrightarrow Y$ is said closable if, for any sequence $(x_n)_{n \geq 1} \subset \mathrm{Dom}(A)$ such that x_n converges to 0 in X and Ax_n converges to y in Y it holds $y = 0$. In other words, A is closable if, for any sequence $(x_n)_{n \geq 1} \subset \mathrm{Dom}(A)$ such that x_n converges to 0 in X and $(x_n)_{n \geq 1}$ is Cauchy w.r.t. the graph norm $\|\cdot\|_G$ it holds Ax_n converges to 0 in Y. The minimal closed extension of the closable operator A is the closed operator $\overline{A}$ whose domain $\mathrm{Dom}(\overline{A})$ is the completion of $\mathrm{Dom}(A)$ w.r.t. $\|\cdot\|_G$, i.e.

$$\mathrm{Dom}(\overline{A}) := \{x \in X : \exists (x_n)_{n \geq 1} \subset \mathrm{Dom}(A) : x_n \to x \text{ in } X$$
$$\text{and } (Ax_n)_{n \geq 1} \text{ converges in } Y\}$$

and we define

$$\overline{A}x := \lim_{n \to \infty} Ax_n, \qquad x \in \mathrm{Dom}(\overline{A}),$$

where the limit is in Y and $(x_n)_{n \geq 1}$ is some sequence in $\mathrm{Dom}(A)$ such that x_n converges to x in X and $(Ax_n)_{n \geq 1}$ converges in Y.

Next, we recall some notions of Dirichlet forms theory. We begin with some definitions related to bilinear forms (see [25] for details). Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A}) \longrightarrow \mathbb{R}$ a bilinear form defined on a dense subspace $\mathrm{Dom}(\mathcal{A})$ of H, the domain of $\mathcal{A}$. The form $\mathcal{A}$ is said to be symmetric if $\mathcal{A}(F, G) = \mathcal{A}(G, F)$, for any $F, G \in \mathrm{Dom}(\mathcal{A})$, and non-negative definite if $\mathcal{A}(F, F) \geq 0$, for any $F \in \mathrm{Dom}(\mathcal{A})$. Let $\mathcal{A}$ be symmetric and non-negative definite, $\mathcal{A}$ is said closed if $\mathrm{Dom}(\mathcal{A})$ equipped with the norm

$$\|F\|_\mathcal{A} := \sqrt{\mathcal{A}(F, F) + \langle F, F \rangle}, \quad F \in \mathrm{Dom}(\mathcal{A}),$$

is a Hilbert space. A symmetric and non-negative definite bilinear form $\mathcal{A}$ is said closable if, for any sequence $(F_n)_{n \geq 1} \subset \mathrm{Dom}(\mathcal{A})$ such that F_n goes to 0 in H and $(F_n)_{n \geq 1}$ is Cauchy w.r.t. $\|\cdot\|_\mathcal{A}$ it holds that $\mathcal{A}(F_n, F_n)$ converges to 0 in $\mathbb{R}$ as n goes to infinity. Let $\mathcal{A}$ be closable and denote by $\mathrm{Dom}(\overline{\mathcal{A}})$ the completion of $\mathrm{Dom}(\mathcal{A})$ w.r.t. the norm $\|\cdot\|_\mathcal{A}$. It turns out that $\mathcal{A}$ is uniquely extended to $\mathrm{Dom}(\overline{\mathcal{A}})$ by the closed, symmetric and non-negative definite bilinear form

$$\overline{\mathcal{A}}(F, G) = \lim_{n \to \infty} \mathcal{A}(F_n, G_n), \quad (F, G) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}}),$$

where $\{(F_n, G_n)\}_{n \geq 1}$ is any sequence in $\mathrm{Dom}(\mathcal{A}) \times \mathrm{Dom}(\mathcal{A})$ such that (F_n, G_n) converges to $(F, G) \in \mathrm{Dom}(\overline{\mathcal{A}}) \times \mathrm{Dom}(\overline{\mathcal{A}})$ w.r.t. the norm $\|\cdot\|_{\overline{\mathcal{A}}} + \|\cdot\|_{\overline{\mathcal{A}}}$. Suppose

$H = L^2(B, \mathcal{B}, \beta)$ where $(B, \mathcal{B}, \beta)$ is a measure space. A symmetric, non-negative definite and closed form $\mathcal{A}$ is said to be a symmetric Dirichlet form if

$$\mathcal{A}(F^+ \wedge 1, F^+ \wedge 1) \leq \mathcal{A}(F, F), \qquad F \in \mathrm{Dom}(\mathcal{A}),$$

where F^+ denotes the positive part of F. Suppose that B is a Hausdorff topological space and let $\mathcal{A}$ be a symmetric Dirichlet form. An $\mathcal{A}$-nest is an increasing sequence $(C_n)_{n \geq 1}$ of closed subsets of B such that

$$\bigcup_{n \geq 1} \{F \in \mathrm{Dom}(\mathcal{A}) \colon F = 0 \ \beta\text{-a.e. on } B \setminus C_n\}$$

is dense in $\mathrm{Dom}(\mathcal{A})$ *w.r.t.* the norm $\|\cdot\|_{\mathcal{A}}$. We say that a subset $B' \subset B$ is $\mathcal{A}$-exceptional if there exists an $\mathcal{A}$-nest $(C_n)_{n \geq 1}$ with $B' \subset B \setminus \bigcup_{n \geq 1} C_n$. Throughout this paper we say that a property holds $\mathcal{A}$-almost everywhere ($\mathcal{A}$-a.e.) if it holds up to an $\mathcal{A}$-exceptional set. Moreover, a function $f : B \to \mathbb{R}$ is called $\mathcal{A}$-almost continuous ($\mathcal{A}$-a.c.) if there exists an $\mathcal{A}$-nest $(C_n)_{n \geq 1}$ such that the restriction $f_{|C_n}$ of f to C_n is continuous for each $n \geq 1$.

Let B be again a Hausdorff topological space. A symmetric Dirichlet form $\mathcal{A}$ on the Hilbert space $L^2(B, \mathscr{S}(B), \beta)$ is called quasi-regular if

(i) There exists an $\mathcal{A}$-nest $(C_n)_{n \geq 1}$ consisting of compact sets.

(ii) There exists a $\|\cdot\|_{\mathcal{A}}$-dense subset of $\mathrm{Dom}(\mathcal{A})$ whose elements have $\mathcal{A}$-a.c. β-versions.

(iii) There exist $F_k \in \mathrm{Dom}(\mathcal{A})$, $k \geq 1$, having $\mathcal{A}$-a.c. β-versions $\tilde{F}_k$, $k \geq 1$, such that $(\tilde{F}_k)_{k \geq 1}$ is a separating set for $B \setminus N$ (i.e. for any $x, y \in B \setminus N$, $x \neq y$, there exists $\tilde{F}_{k^*}$ such that $\tilde{F}_{k^*}(x) \neq \tilde{F}_{k^*}(y)$), where N is a subset of B which is $\mathcal{A}$-exceptional.

Acknowledgements

References

1. S. Albeverio, Y. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154: 444–500, 1998.

2. A. Borodin, A. Okounkov and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups *J. Amer. Math. Soc.*, 13: 481–515, 2000.

3. A. Borodin and G. Olshanski. Point processes and the infinite symmetric group. *Math. Res. Let.*, 5: 799–816, 1998.

4. A. Borodin and G. Olshanski. Distributions on partitions, point processes, and the hypergeometric kernel. *Comm. Math. Phys.*, 211: 335–358, 2000.

5. A. Borodin and G. Olshanski. Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes. *Ann. of Math.*, 161: 1319–1422, 2005.

6. A. Borodin and G. Olshanski. Random partitions and the gamma kernel. *Adv. Math.*, 194: 141–202, 2005.

7. H. Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

8. I. Camilier and L. Decreusefond. Quasi-invariance and integration by parts for determinantal and permanental processes. *J. Funct. Anal.*, 259: 268–300, 2010.

9. D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2003. Elementary theory and methods.

10. D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II.* Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

11. L. Decreusefond, I. Flint, N. Privault, and G. L. Torrisi. Stochastic dynamics of determinantal processes by integration by parts. arXiv:1210.6109v3, 2014.

12. L. Decreusefond, I. Flint, and A. Vergne. Efficient simulation of the Ginibre point process. *J. Appl. Probab.*, to appear, 2015.

13. N. Dunford and J. T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

14. M. Fukushima, Y. Ōshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics.* Walter de Gruyter & Co., Berlin, 1994.

15. J. Ginibre. Statistical ensembles of complex, quaternion, and real matrices. *J. Mathematical Phys.*, 6: 440–449, 1965.

16. H. Georgii and H. J. Yoo. Conditional intensity and Gibbsianness of determinantal point processes. *J. Stat. Phys.*, 118: 55–84, 2005.

17. J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág. Determinantal processes and independence. *Probab. Surv.*, 3:206–229 (electronic), 2006.

18. J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2009.

19. W. S. Kendall. Perfect simulation for the area-interaction point process. In *Probability towards 2000 (New York, 1995)*, volume 128 of *Lecture Notes in Statist.*, pages 218–234. Springer, New York, 1998.

20. W. S. Kendall and J. Møller. Perfect simulation using dominating processes on ordered spaces, with application to locally stable point processes. *Adv. in Appl. Probab.*, 32: 844–865, 2000.

21. G. Last. Stochastic analysis for Poisson processes. In this volume, 2015.

22. F. Lavancier, J. Møller, and E. Rubak. Determinantal point process models and statistical inference. *Journal of the Royal Statistical Society: Series B*, 77: 853–877, 2015.

23. M. N. M. van Lieshout. *Markov point processes and their applications.* Imperial College Press, London, 2000.

24. E. Lytvynov. Determinantal point processes with J-Hermitian correlation kernels. *Ann. Probab.*, 41: 2513–2543, 2013.

25. Z.M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms.* Universitext. Springer-Verlag, Berlin, 1992.

26. Z.M. Ma and M. Röckner. Construction of diffusions on configuration spaces. *Osaka J. Math.*, 37: 273–314, 2000.

27. O. Macchi. The coincidence approach to stochastic point processes. *Advances in Applied Probability*, 7: 83–122, 1975.

28. N. Miyoshi and T. Shirai. Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations. *Performance Evaluation*, 77: 7–17, 2014.

29. J. Møller and R. P. Waagepetersen. *Statistical inference and simulation for spatial point processes*, volume 100 of *Monographs on Statistics and Applied Probability.* Chapman & Hall/CRC, Boca Raton, FL, 2004.

30. P. Robert. *Stochastic networks and queues*, Stochastic Modelling and Applied Probability Series, Springer-Verlag, New York, 2003.

31. T. Shirai. Large deviations for the fermion point process associated with the exponential kernel *J. Stat. Phys.*, 123: 615-629, 2006.

32. T. Shirai and Y. Takahashi. Random point fields associated with certain Fredholm determinants I. fermion, Poisson and boson point processes. *J. of Functional Analysis*, 205: 414–463, 2003.

33. B. Simon. *Trace ideals and their applications*, volume 120 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2005.

34. A. Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55: 107–160, 2000.

35. G.L. Torrisi and E. Leonardi. Large deviations of the interference in the Ginibre network model. *Stochastic Systems*, 4: 173-205, 2014.

36. G.L. Torrisi. Quasi-invariance of fermion processes with J-Hermitian kernel. *Communications on Stochastic Analysis*, 8: 169–190, 2014.

37. G.L. Torrisi. Point processes with Papangelou conditional intensity: from the Skorohod integral to the Dirichlet form. *Markov Processes and Related Fields*, 19:195–248, 2013.