Rules for Computing Resistance of Transitions of Learning Algorithms in Games

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Abstract. In a finite game the Stochastically Stable States (SSSs) of adaptive play are contained in the set of minimizers of resistance trees. Also, in potential games, the SSSs of the log-linear learning algorithm are the minimizers of the potential function. The SSSs can be characterized using the resistance trees of a Perturbed Markov Chain (PMC), they are the roots of minimum resistance tree. Therefore, computing the resistance of trees in PMC is important to analyze the SSSs of learning algorithms. A learning algorithm defines the Transition Probability Function (TPF) of the induced PMC on the action space of the game. Depending on the characteristics of the algorithm the TPF may become composite and intricate. Resistance computation of intricate functions is difficult and may even be infeasible. Moreover, there are no rules or tools available to simplify the resistance computations. In this paper, we propose novel rules that simplify the computation of resistance. We first, give a generalized definition of resistance that allows us to overcome the limitations of the existing definition. Then, using this new definition we develop the rules that reduce the resistance computation of composite TPF into resistance computation of simple functions. We illustrate their strength by efficiently computing the resistance in log-linear and payoff-based learning algorithms. They provide an efficient tool for characterizing SSSs of learning algorithms in finite games.

Key words: potential games, learning algorithms, log-linear learning, perturbed markov chains, resistance of transitions

1 Introduction

In a finite repeated game if players sometimes make mistakes in choosing an optimal strategy and if all mistakes are possible and are time-independent then a perturbed Markov process is induced on the action space of the game. As the probability of mistakes goes to zero the stationary distribution of the process concentrates on particular equilibria. These are known as stochastically stable equilibria or Stochastically Stable States (SSS) of the game [1]. The SSSs correspond to the roots of minimum resistance trees where the resistance of a transition in a tree can be seen as the cost of deviating from the optimal strategy [2]. Therefore, the computation of resistance of transitions of a Perturbed Markov Chain (PMC) is important.

The learning algorithm used by the player of the game defines the Transition Probability Function (TPF) of the induced PMC. Depending on the characteristics of the learning algorithm the TPF can be composite and intricate. The resistance computation of intricate TPF is difficult and may even be infeasible for some functions. Moreover, there are no rules and no tools available in the literature to simplify the computation of resistance. We focus on developing novel rules to simplify the resistance computations of a general class of TPF.

As the perturbation slowly decreases the limiting stationary distribution of a PMC exists and is unique [2]. The support of the stationary distribution is the root of the minimum resistance tree. Exploring these results many learning algorithms for games are analyzed in the literature. In the following, we discuss a few such algorithms.

Log-linear learning algorithm is used for potential games that models the load balancing problem of a heterogeneous wireless network [3]. In this algorithm, the log of TPF is linear functions of the payoffs of the players [3–5]. This algorithm induces a PMC on the action space of the game. The convergence of this algorithms is analyzed as follows. First, using the TPF in (5) [5] the expression of resistance of transition is (6) [5] is obtained. We observe that the derivation of this expression requires a careful insight into the TPF to reduce it into a simplified form so that the resistance can be obtained. Otherwise, in case the TPT cannot be reduced into a simple form then the resistance may not be feasible to compute. Second, the resistance of a feasible path in a tree is obtained using the structure of potential games. Finally, the SSSs of the game are characterized by using the minimum resistance tree definition. Binary log-linear learning algorithm is a reduced information algorithm, in which the log TPF is linear function of the two most recent payoffs [5]. This algorithm was used to distributively balance the loads in heterogeneous networks using near-potential games [6]. The computation resistance of transition is difficult in this case. The convergence of this algorithm to the SSSs of a potential game is analyzed in the similar way as in log-linear algorithm [5, 6].

A payoff-based learning algorithm is obtained by combining log-linear algorithm and binary log-linear algorithm [5]. Due to the combination of two algorithms, the TPF is much involved. Therefore, the computation of resistance of transition is much involved and difficult. The convergence of this algorithm is also analyzed in a similar way as in log-linear algorithm. Adaptive play algorithm was applied to acyclic game to characterize its SSSs using the resistance trees [2]. A class of trial and error learning algorithms for any finite game are also analyzed using the resistance trees [7, 8]. Due to the different modes of learning in these algorithms the TPF becomes complicated and the resistance computation is difficult.

In the above literature survey, we see that the computation of resistance are used for characterizing the SSSs of many learning algorithms in games. Therefore, in this paper, we develop new rules that ease the computation of resistance of intricate TPF. To do this, we first give a generalized definition of resistance for any positive function. The new definition overcomes the limitation of the existing old definition of resistance. For example, the limit in the old definition of resistance is not always be feasible to evaluate for some functions, see Section 3. The new definition allows us to define resistance for any positive function. Thereby, allowing us to propose new rules for computing resistance. The proposed rules reduce the resistance computation of composite TPF into resistance computation of simple functions. These rules provide a powerful tool that can be used for analyzing the convergence properties of learning algorithms in finite games.

The rest of the paper is organized as follows. In Section 2, we give an overview of resistance trees of PMC. In Section 3, we present new rules for resistance computation and provide their proves. In Section 4, we illustrate the application of the proposed rules. Conclusions are summarized in Section 5.

2 Overview of Resistance Trees

In this section, we first give a brief overview of resistance trees of a PMC. Then, using resistance trees we illustrate the convergence of log-linear learning algorithm in potential games. For more details see [2, 5].

2.1 Resistance trees of PMC

A perturbed Markov process is characterized by a set $\{P^\tau\}$ of transition matrices over a state space X indexed by a parameter τ. Wherein, $\tau \in (0, \tau_h]$ is a parameter that controls the perturbation, τ_h is constant. Probabilities P^0_{ab} and P^τ_{ab} denote the transition probabilities from state a to b in the unperturbed and the perturbed Markov chains, respectively. The definition of resistance of transitions and the definition of a regular perturbed Markov process are below [2].

Definition 1 (Resistance of transition). *A perturbed Markov process $\{P^\tau\}$ is a regular if it satisfies the following conditions [2]:*

1. *P^τ is aperiodic and irreducible for all $\tau \in (0, \tau_h]$,*
2. $\lim_{\tau \to 0} P^\tau_{ab} = P^0_{ab}$,
3. *for a strictly positive TPF P^τ_{ab} there exists a non-negative number R_{ab} called the resistance of transition such that* $0 < \lim_{\tau \to 0^+} e^{\frac{R_{ab}}{\tau}} P^\tau_{ab} < \infty$.

Note that if $P^0_{ab} > 0$ then $R_{\text{ab}} = 0$.

A tree, T, rooted at a state a, is a set of $|X| - 1$ directed edges such that, from every other state a', there is a unique directed path in the tree to a. The resistance of the directed edge $a \to b$ is denoted as R_{ab}. The resistance of a rooted tree, T, is the sum of the resistances on its edges $R(T) = \sum_{a,b \in T} R_{\text{ab}}$. Let $\mathcal{T}(a)$ be defined as the set of trees rooted at the state a. The stochastic potential of the state a is defined as $\gamma(a) = \min_{T \in \mathcal{T}(a)} R(T)$. A minimum resistance tree is a tree that has the minimum stochastic potential, that is, any tree T that satisfies $R(T) = \min_{a \in X} \gamma(a)$.

The following theorem by [2, Lemma 1] gives the existence and uniqueness of the stationary distribution of a PMC.

Theorem 1. *Let $\{P^\tau\}$ be a regular perturbed Markov process, and for each $\tau > 0$, let μ_τ be the unique stationary distribution of P^τ. Then $\lim_{\tau \to 0} \mu_\tau$ exists and the limiting distribution μ_0 is a stationary distribution of P^0. The stochastically stable states are the roots of minimum resistance trees.*

2.2 Convergence of log-linear learning algorithm using resistance trees [5]

Log-linear learning algorithm induces a regular perturbed Markov process over the action space X of a n-player potential game [5]. Let $a = (a_i, a_{-i})$ denotes an action profile of the players where a_i denote the action of player i and a_{-i} denotes the actions of all the other players. Let X_i and X_{-i} denote the action space of player i and action space of other players, respectively. Let $b = (a_i', a_{-i})$ denotes another action profile where player i changes its action. For $a \in X$, let $\phi(a)$ and $U_i(a)$ denote the potential function and utility of player i, respectively. In a potential game, for all $a_i, a_i' \in X_i$ and for all $a_{-i} \in X_{-i}$, we have $\phi(a) - \phi(b) = U_i(a) - U_i(b)$. Assuming that the player is selected with uniform probability the transition probability function of log-linear learning algorithm is given as below [5, (5)].

$$P_{ab}^{\tau} = \frac{1}{n} \frac{\exp\left(\frac{U_i(a_i', a_{-i})}{\tau}\right)}{\sum_{a_i \in X_i} \exp\left(\frac{U_i(a_i, a_{-i})}{\tau}\right)} \tag{1}$$

The first step in the proof of convergence is to derive an expression of resistance of transition. Let $V(a_{-i}) := \max_{a_i \in X_i} U_i(a_i, a_{-i})$ and $B_i(a_i)$ denotes the set of actions that have the maximum utility. Multiplying the numerator and denominator of (1) by $e^{\frac{V(a_{-i})}{\tau}}$, we obtain

$$P_{ab}^{\tau} = \frac{1}{n} \frac{\exp\left(\frac{V(a_{-i}) - U_i(a_i', a_{-i})}{\tau}\right)}{\sum_{a_i \in X_i} \exp\left(\frac{V(a_{-i}) - U_i(a_i, a_{-i})}{\tau}\right)}. \tag{2}$$

After simplifying the above equation, we obtain

$$\lim_{\tau \to 0^+} \frac{P_{ab}^{\tau}}{\exp\left(\frac{V(a_{-i}) - U_i(a_i', a_{-i})}{\tau}\right)} = \frac{1}{n \left|B_i(a_i)\right|}. \tag{3}$$

Since, the above limit is positive and finite the induced process is a regular Markov process and the resistance according to Definition 1 is

$$R_{ab} = V(a_{-i}) - U_i(a_i', a_{-i}). \tag{4}$$

Second step is to obtain the resistance of a path in the resistance trees. This is obtained in Lemma [5, Lemma 3.2] that we present below.

Lemma 1. *Let $\mathcal{P} = \left\{a^0 \to a^1 \to \ldots \to a^m\right\}$ and $\mathcal{P}^R = \left\{a^m \to a^{m-1} \to \ldots \to a^0\right\}$ be feasible forward path and reverse path, respectively. If all the players in a n-player potential game with potential function $\phi : X \to \mathcal{R}$, adhere to log-linear learning algorithm then the difference of resistance of paths is*

$$R(\mathcal{P}) - R\left(\mathcal{P}^R\right) = \phi\left(a^0\right) - \phi\left(a^m\right). \tag{5}$$

The final step is to prove that the stochastically stable states of the log-linear algorithm are the potential function maximizers of the potential game. This is accomplished by using Lemma 1 and minimum resistance tree definition. The detailed proof of the following theorem can be found in Proposition [5, 3.1].

Theorem 2. *If all the players of a potential game adhere to log-linear learning algorithm then the stochastically stable states are the potential function maximizers.*

3 Rules for Computing Resistance

The resistance in Definition 1 can be computed in case the transition function can be factorised into simple function and in case the limit can be evaluated as shown in Section 2.2. However, transition functions can be composite and intricate that cannot always be simplified. Moreover, the limit in Definition 1 cannot always be feasible to evaluate. For example, when $P_{ab}^{\tau} = \tau$, the limit cannot be evaluated. To overcome these limitations of Definition 1 we first give a new generalised definition of resistance that allows us to develop easy rules to compute the resistance of any positive function.

Let $o\left(.\right)$ and $\omega\left(.\right)$ denote little "o" order and little "ω" order, respectively.

Definition 2 (Resistance of positive function). *The resistance of a strictly positive function $f(\tau)$ is Res(f) if there exists a strictly positive function $g(\tau)$ such that $g \in o\left(e^{k/\tau}\right)$ and $g \in \omega\left(e^{-k/\tau}\right)$ for any $k > 0$; and*

$$\lim_{\tau \to 0} \frac{f(\tau)}{g(\tau)e^{-\frac{Res(f)}{\tau}}} = 1. \tag{6}$$

Remark 1. Note that Definition 2 includes Definition 1, in which $g(\tau) = \kappa, 0 < \kappa < \infty$. Now, we can evaluate the resistance of $P_{ab}^{\tau} = \tau$, i.e., $\text{Res}(\tau) = 0$.

Remark 2. Note that (6) is equivalent to

$$f(\tau) = g(\tau)e^{-\frac{\text{Res}(f)}{\tau}} + h(\tau), \tag{7}$$

where $h(\tau) \in o\left(g(\tau)e^{-\frac{\text{Res}(f)}{\tau}}\right)$.

Remark 3. We call $g(\tau)$ as a *sub-exponential function* if $g \in o\left(e^{k/\tau}\right)$ and $g \in \omega\left(e^{-k/\tau}\right)$ for any $k > 0$. Note that it is equivalent to $|\log g| \in o\left(\frac{1}{\tau}\right)$.

Lemma 2. *Consider any two sub-exponential functions $g_1(\tau)$ and $g_2(\tau)$. Consider two real numbers R_1 and R_2. If $R_1 < R_2$ then*

$$g_2(\tau)e^{-R_2/\tau} \in o\left(g_1(\tau)e^{-R_1/\tau}\right). \tag{8}$$

Proof. Let k be a real number. Then

$$\lim_{\tau \to 0} \frac{g_2(\tau)e^{-R_2/\tau}}{g_1(\tau)e^{-R_1/\tau}} = \lim_{\tau \to 0} \frac{g_2(\tau)}{e^{-(R_2-k)/\tau}} \left[\frac{g_1(\tau)}{e^{-(R_1-k)/\tau}}\right]^{-1}. \tag{9}$$

The above limit goes to zero when we choose $R_1 < k < R_2$. This is because the first factor goes to zero as $R_2 - k > 0$. Also, the second factor goes to zero as $R_1 - k < 0$. Recall that it is because g_1 and g_2 are sub-exponential.

Lemma 3. *If Res(f) exists then it is unique.*

Proof. Assume that function f have two different resistances R_1 and R_2. Then, there exist g_1, g_2, h_1, h_2 such that

$$f(\tau) = g_1(\tau)e^{-\frac{R_1}{\tau}} + h_1(\tau) = g_2(\tau)e^{-\frac{R_2}{\tau}} + h_2(\tau), \tag{10}$$

where $h_1(\tau) \in o\left(g_1(\tau)e^{-\frac{R_1}{\tau}}\right)$ and $h_2(\tau) \in o\left(g_2(\tau)e^{-\frac{R_2}{\tau}}\right)$. Let $R_1 < R_2$. Using Lemma 2, we have $h_2 \in o\left(g_1(\tau)e^{-\frac{R_1}{\tau}}\right)$. Rearranging terms in (10), we have

$$1 + \frac{h_1(\tau)}{g_1(\tau)e^{-\frac{R_1}{\tau}}} = \frac{g_2(\tau)e^{-\frac{R_2}{\tau}}}{g_1(\tau)e^{-\frac{R_1}{\tau}}} + \frac{h_2(\tau)}{g_1(\tau)e^{-\frac{R_1}{\tau}}}. \tag{11}$$

Using Lemma 2 to evaluate the limit of the above equation as τ goes to zero, we arrive at contradiction that $1 = 0$.

The following proposition gives the rules for computing Res(f).

Proposition 1. *Let f, f_1 and f_2 be strictly positive functions. Let κ be a positive constant. If Res(f_1) and Res(f_2) exist then*

- *I $f_1(\tau)$ is sub-exponential if and only if Res$(f_1) = 0$. In particular Res$(\kappa) = 0$,*
- *II Res$(e^{-\kappa/\tau}) = \kappa$,*
- *III Res$(f_1 + f_2) = \min\{$Res(f_1), Res$(f_2)\}$,*
- *IV Res$(f_1 - f_2) =$ Res(f_1), if Res$(f_1) <$ Res(f_2),*
- *V Res$(f_1 f_2) =$ Res$(f_1) +$ Res(f_2),*
- *VI Res$(\frac{1}{f}) = -$Res(f),*
- *VII If $f_1(\tau) \le f_2(\tau)$, Res(f_1) and Res(f_2) exist then Res$(f_2) \le$ Res(f_1),*
- *VIII Let $f_1(\tau) \le f(\tau) \le f_2(\tau)$, If Res$(f_1) =$ Res(f_2) then Res(f) exists and Res$(f) =$ Res(f_1).*

Remark 4. In Rule IV, if Res$(f_1) =$ Res(f_2) then we cannot compute Res$(f_1 - f_2)$ because in general the difference of sub-exponential functions may not be a sub-exponential function. For example, choose $f_1(\tau) = 1 + e^{-k/\tau}$ and $f_2(\tau) = 1$ with $k > 0$ then Res$(f_1) =$ Res$(f_2) = 0$ but Res$(f_1 - f_2) = k$.

Remark 5. For Rule VIII, in general if $f_1(\tau) \le f(\tau) \le f_2(\tau)$ and Res$(f_1) \ne$ Res(f_2) then Res(f) may not exist. For example, for $f(\tau) = \lambda(\tau)f_1 + (1 - \lambda(\tau))f_2$, $\lambda(\tau) = \frac{1}{2}\left(\cos\left(\frac{1}{\tau}\right) + 1\right)$ the Res(f) does not exist.

Proof. Proof of Rule I: Let $f(\tau)$ be a sub-exponential function. Choosing $g(\tau) = f(\tau)$ and substituting Res$(f) = 0$ in (6) we get $\lim_{\tau\to 0} \frac{f(\tau)}{f(\tau)e^{-\frac{\text{Res}(f)}{\tau}}} = 1$. Therefore, we have Res$(f) = 0$.

Assume Res$(f) = 0$. From (7), we have $f(\tau) = g(\tau) + h(\tau)$, which is a sub-exponential function.

Let $f(\tau) = \kappa$ and $g(\tau) = \kappa$ then $g(\tau) \in o\left(e^{\frac{\kappa}{\tau}}\right)$ and $g(\tau) \in \omega\left(e^{-\frac{\kappa}{\tau}}\right)$, $\kappa > 0$. Substituting these in (6) we have Res$(\kappa) = 0$.

Proof of Rule II: Substituting $f(\tau) = e^{-\kappa/\tau}$ and $g(\tau) = 1$ in (6) we get $\text{Res}(f) = \kappa$.

Proof of Rule III: Let $\text{Res}(f_1)$ and $\text{Res}(f_2)$ be the resistances of functions f_1 and f_2, respectively. Then, from (7) we have $f_1(\tau) = g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}} + h_1(\tau)$, $f_2(\tau) = g_2(\tau)e^{-\frac{\text{Res}(f_2)}{\tau}} + h_2(\tau)$, where $h_1(\tau) \in o\left(g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}\right)$, $h_2(\tau) \in o\left(g_2(\tau)e^{-\frac{\text{Res}(f_2)}{\tau}}\right)$. The sum of two functions can be written as

$$f_1(\tau) + f_2(\tau) = g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}} \left(1 + \frac{h_1(\tau)}{g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}} + \frac{g_2(\tau)e^{-\frac{\text{Res}(f_2)}{\tau}}}{g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}} + \frac{h_2(\tau)}{g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}}\right), \quad (12)$$

Consider the case when $\text{Res}(f_1) < \text{Res}(f_2)$. Using Lemma 2 we have $h_2 \in o\left(g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}\right)$. Therefore, $f_1(\tau) + f_2(\tau) = g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}} + h_3(\tau)$, where $h_3(\tau) \in o\left(g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}\right)$. According to (7), we have $\text{Res}(f_1 + f_2) = \text{Res}(f_1)$.

The case of $\text{Res}(f_1) = \text{Res}(f_2)$ leads to the same result as shown below.

$$f_1(\tau) + f_2(\tau) = e^{-\frac{\text{Res}(f_1)}{\tau}} \left[g_1(\tau) + g_2(\tau)\right] + h_1(\tau) + h_2(\tau). \quad (13)$$

Note that sum of sub-exponential functions $g_1(\tau) + g_2(\tau)$ is a sub-exponential function. Observe that $h_1(\tau) + h_2(\tau) \in o\left(\left[g_1(\tau) + g_2(\tau)\right] e^{-\frac{\text{Res}(f_1)}{\tau}}\right)$. As in the previous case, according to (7) we have $\text{Res}(f_1 + f_2) = \text{Res}(f_1)$

Proof of Rule IV: Also, it can be shown similarly to the proof of rule III that if $\text{Res}(f_1) < \text{Res}(f_2)$ then $\text{Res}(f_1 - f_2) = \text{Res}(f_1)$.

Proof of Rule V:

$$\lim_{\tau\to 0} \frac{f_1(\tau)}{g_1(\tau)e^{-\frac{\text{Res}(f_1)}{\tau}}} \lim_{\tau\to 0} \frac{f_2(\tau)}{g_2(\tau)e^{-\frac{\text{Res}(f_2)}{\tau}}} = \lim_{\tau\to 0} \frac{f_1(\tau)f_2(\tau)}{g_1(\tau)g_2(\tau)e^{-\frac{\text{Res}(f_1)+\text{Res}(f_1)}{\tau}}} = 1. \quad (14)$$

Therefore, $\text{Res}(f_1 f_2) = \text{Res}(f_1) + \text{Res}(f_2)$.

Proof of Rule VI: Since $\text{Res}(f)$ exists, inverting both sides of (6), we have

$$\lim_{\tau\to 0} \frac{f(\tau)}{g(\tau)e^{-\frac{\text{Res}(f)}{\tau}}} = 1 = \lim_{\tau\to 0} \frac{\frac{1}{f(\tau)}}{\frac{1}{g(\tau)}e^{-\frac{-\text{Res}(f)}{\tau}}}. \quad (15)$$

Note that $\frac{1}{g(\tau)}$ is sub-exponential. Therefore, we have $\text{Res}(\frac{1}{f}) = -\text{Res}(f)$.

Proof of Rule VII: Assume that $\text{Res}(f_1) < \text{Res}(f_2)$. Using Lemma 2, we have $g_2(\tau)e^{-\text{Res}(f_2)/\tau} \in o\left(g_1(\tau)e^{-\text{Res}(f_1)/\tau}\right)$ and $h_2 \in o\left(g_1(\tau)e^{-\text{Res}(f_1)/\tau}\right)$.

$$f_1 \leq f_2, \quad (16)$$

$$g_1(\tau)e^{-\text{Res}(f_1)/\tau} + h_1(\tau) \leq g_2(\tau)e^{-\text{Res}(f_2)/\tau} + h_2(\tau), \quad (17)$$

$$1 + \frac{h_1(\tau)}{g_1(\tau)e^{-\text{Res}(f_1)/\tau}} \leq \frac{g_2(\tau)e^{-\text{Res}(f_2)/\tau} + h_2(\tau)}{g_1(\tau)e^{-\text{Res}(f_1)/\tau}}. \quad (18)$$

As $\tau \to 0$, we arrive at a contradiction that $1 \leq 0$. Therefore, $\text{Res}(f_1) \geq \text{Res}(f_2)$.

Proof of Rule VIII: We have $1 \leq \frac{f(\tau)}{f_1(\tau)} \leq \frac{f_2(\tau)}{f_1(\tau)}$ and $\text{Res}\left(\frac{f_2(\tau)}{f_1(\tau)}\right) = \text{Res}(f_2) - \text{Res}(f_1) = 0$. By Rule I $\frac{f_2(\tau)}{f_1(\tau)}$ is sub-exponential. This implies that $\frac{f(\tau)}{f_1(\tau)}$ is also sub-exponential. Therefore, there exists $g_{01}(\tau)$ such that

$$1 = \lim_{\tau\to 0} \frac{\frac{f(\tau)}{f_1(\tau)}}{g_{01}(\tau)} = \lim_{\tau\to 0} \frac{f(\tau)}{g_{01}(\tau) g_1(\tau) e^{-\frac{\text{Res}(f_1)}{\tau}}} \lim_{\tau\to 0} \frac{g_1(\tau) e^{-\frac{\text{Res}(f_1)}{\tau}}}{f_1(\tau)}, \tag{19}$$

$$= \lim_{\tau\to 0} \frac{f(\tau)}{g_{01}(\tau) g_1(\tau) e^{-\frac{\text{Res}(f_1)}{\tau}}}, \tag{20}$$

where the product $g_{01}(\tau)g_1(\tau)$ is also a sub-exponential function. Therefore, $\text{Res}(f)$ exists and $\text{Res}(f) = \text{Res}(f_1) = \text{Res}(f_2)$.

4 Application of Proposed Rules

In this section, we illustrate the application and robustness of the proposed rules for computing the resistance of composite TPFs.

4.1 Resistance of log-linear learning algorithm

By using Rule V and VI the resistance of $\text{Res}\left(P_{ab}^{\tau}\right)$ (1) is obtained as below.

$$\text{Res}\left(P_{ab}^{\tau}\right) = \text{Res}\left(\frac{1}{n}\right) + \text{Res}\left(e^{\frac{U_i(a_i', a_{-i})}{\tau}}\right) - \text{Res}\left(\sum_{a_i \in X_i} e^{\frac{U_i(a_i, a_{-i})}{\tau}}\right). \tag{21}$$

Applying the Rule III to the above equation, we have

$$\text{Res}\left(P_{ab}^{\tau}\right) = \text{Res}\left(\frac{1}{n}\right) + \text{Res}\left(e^{\frac{U_i(a_i', a_{-i})}{\tau}}\right) - \min_{a_i \in X_i} \text{Res}\left(e^{\frac{U_i(a_i, a_{-i})}{\tau}}\right). \tag{22}$$

Applying the Rule I and II, we get

$$\text{Res}\left(P_{ab}^{\tau}\right) = -U_i(a_i', a_{-i}) - \min_{a_i \in X_i} \left(-U_i(a_i, a_{-i})\right) = V\left(a_{-i}\right) - U_i(a_i', a_{-i}). \tag{23}$$

4.2 Resistance of payoff-based learning algorithm

In this subsection, we illustrate the application of the proposed rules by obtaining the expression of resistance payoff-based algorithm as in [5, Claim 6.1]. Let denotes two states of PMC of this algorithm as $z^1 := \left[a^0, a^1, x^1\right]$ and $z^2 := \left[a^1, a^2, x^2\right]$, where a^0, a^1, a^2 are action profiles and x^1, x^2 denotes the vectors representing whether the players have experimented or not, $x_i^1 = 0$ and $x_i^2 = 1$ represents that the player i had experimented. The transition probability function of Payoff-based algorithm is much involved as can be seen in [5, Claim 6.1].

$$P^{\tau}_{z^1 \to z^2} = \left(\prod_{i:x_i^1=0, x_i^2=0} \left(1 - e^{-\frac{m}{\tau}}\right) \right) \left(\prod_{i:x_i^1=0, x_i^2=1} \frac{e^{-\frac{m}{\tau}}}{|X_i|} \right)$$
$$\left(\prod_{i:x_i^1=1, a_i^2=a_i^0} \frac{e^{\frac{U_i(a^0)}{\tau}}}{e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}} \right) \left(\prod_{i:x_i^1=1, a_i^2=a_i^1} \frac{e^{\frac{U_i(a^1)}{\tau}}}{e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}} \right) \quad (24)$$

Using the Rule V, we have

$$\text{Res}\left(P^{\tau}_{z^1 \to z^2}\right) = \sum_{i:x_i^1=0, x_i^2=0} \text{Res}\left(1 - e^{-\frac{m}{\tau}}\right) + \sum_{i:x_i^1=0, x_i^2=1} \text{Res}\left(\frac{e^{-\frac{m}{\tau}}}{|X_i|}\right)$$
$$\sum_{i:x_i^1=1, a_i^2=a_i^0} \text{Res}\left(\frac{e^{\frac{U_i(a^0)}{\tau}}}{e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}}\right) + \sum_{i:x_i^1=1, a_i^2=a_i^1} \text{Res}\left(\frac{e^{\frac{U_i(a^1)}{\tau}}}{e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}}\right) \quad (25)$$

Applying the Rules III, IV, V, and VI, we have

$$\text{Res}\left(P^{\tau}_{z^1 \to z^2}\right) = \sum_{i:x_i^1=0, x_i^2=0} \min\left\{\text{Res}\,(1), \text{Res}\left(e^{-\frac{m}{\tau}}\right)\right\}$$
$$+ \sum_{i:x_i^1=0, x_i^2=1} \left[\text{Res}\left(e^{-\frac{m}{\tau}}\right) + \text{Res}\left(\frac{1}{|X_i|}\right)\right]$$
$$+ \sum_{i:x_i^1=1, a_i^2=a_i^0} \left[\text{Res}\left(e^{\frac{U_i(a^0)}{\tau}}\right) - \text{Res}\left(e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}\right)\right]$$
$$+ \sum_{i:x_i^1=1, a_i^2=a_i^1} \left[\text{Res}\left(e^{\frac{U_i(a^1)}{\tau}}\right) - \text{Res}\left(e^{\frac{U_i(a^0)}{\tau}} + e^{\frac{U_i(a^1)}{\tau}}\right)\right] \quad (26)$$

Simplifying further by applying the Rules I and II, we get

$$\text{Res}\left(P^{\tau}_{z^1 \to z^2}\right) = \sum_{i:x_i^1=0, x_i^2=0} \min\{0, m\} + \sum_{i:x_i^1=0, x_i^2=1} [m]$$
$$+ \sum_{i:x_i^1=1, a_i^2=a_i^0} \left[-U_i(a^0) - \min\left\{-U_i(a^0), -U_i(a^1)\right\}\right]$$
$$+ \sum_{i:x_i^1=1, a_i^2=a_i^1} \left[-U_i(a^1) - \min\left\{-U_i(a^0), -U_i(a^1)\right\}\right] \quad (27)$$

Let $V(a^0, a^1) = \max\left\{U_i(a^1), U_i(a^2)\right\}$, then we have

$$\text{Res}\left(P^{\tau}_{z^1 \to z^2}\right) = \sum_{i:x_i^1=0,x_i^2=1} m + \sum_{i:x_i^1=1,a_i^2=a_i^0} \left(V(a^0,a^1) - U_i(a^0)\right) + \sum_{i:x_i^1=1,a_i^2=a_i^1} \left(V(a^0,a^1) - U_i(a^1)\right) \quad (28)$$

The above obtained expression of resistance is same as in [5, (13)], verifying it.

5 Conclusion

Novel rules are proposed for computing the resistance of transition of a perturbed Markov chain. These rules reduce the computation of resistance of composite and intricate transition probability function into the computation of resistance of simple functions. These rules are simple and yet are powerful. The strength of these rules is illustrated by using them to calculate efficiently the resistance of transition of the well-known log-linear learning algorithm and the payoff-based learning algorithm. These calculations are verified by comparing the obtained expressions with that of in the literature. These rules provide an efficient tool that can be used to characterize the stochastically stable states of learning algorithms in finite games. We hope to apply these rules for analyzing new algorithms based on perturbed Markov chains as well as new game settings like potential games with noisy rewards [9].

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