Stochastic Integration with respect to Volterra processes

L. Decreusefond

Abstract

We construct the basis of a stochastic calculus for so-called Volterra processes, i.e., processes which are defined as the stochastic integral of a time-dependent kernel with respect to a standard Brownian motion. For these processes which are natural generalization of fractional Brownian motion, we construct a stochastic integral and show some of its main properties: regularity with respect to time and kernel, transformation under an absolutely continuous change of probability, possible approximation schemes and Itô formula.

1 Introduction

In the past few years, more than twenty papers have been devoted to the definition of a stochastic integral with respect to fractional Brownian motion or other "related" processes, see for instance [Dec02a] and references therein. Remind that fractional Brownian process of Hurst index $H \in (0,1)$, denoted by B^H, is the unique centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = \mathrm{E}\left[B_s^H B_t^H\right] \stackrel{def}{=} \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right)$$

where

$$V_H \stackrel{def}{=} \frac{\Gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}.$$

Among other properties, this process has $1/H$-finite variation and a finite generalized covariation of order 4 for $H > 1/4$, (see [GRV] for the definition), has Hölder continuous trajectories of any order less than H and has the following representation property:

$$B^H(t) = \int_0^t K_H(t,s)\, \mathrm{d}B_s, \tag{1}$$

where B is a one dimensional standard Brownian motion and K is deterministic kernel with an intricate expression (see [DÜ99]). Therefore, a "related" process means altogether a process with finite p-variation, called a process with rough paths in [CQ00, Lyo98], or a process with Hölder continuous sample-paths as in [FdLP99, Z̈98] and also a process of the form (1) with a general kernel as in [AMN01, CCM02, Dec02b].

This is the last track that we will follow here. Our present work, which is the expanded version of [Dec02b], differs from the other two papers [AMN01, CCM02] in two ways. First, the method to define the stochastic integral is different. In these two papers, the kernel is regularized, if needed, to obtain a semi-martingale. The second step is then to use the classical theory of stochastic integration and then pass to the limit after a stochastic integration by parts in the sense of the Malliavin Calculus. We here use an approach based on convergence of discrete sums. It should be already noted that for smooth integrands, their notion of integral and ours coincide. The other difference is to be found in the kind of hypothesis put on K. In [AMN01, CCM02], hypothesis are made on the regularity of the function $K(t,s)$ itself. We here work with assumptions on the linear map $f \mapsto \int K(t,s) f(s)\, \mathrm{d}s$. Properties of $K(t,s)$ and Kf are, of course, intimately related but we think that working with the latter gives more insight on the underlying problems.

In Section 2, we recall basic definitions and properties of deterministic fractional calculus. In Section 3, we introduce the class of processes, named Volterrra processes, that we will study. We then give a few properties of their sample-paths. In Section 4, we deal with a Stratonovitch-like definition of the stochastic integral with respect to Volterra processes. Section 5 is devoted to the time regularity of the previously constructed integral and in Section 6, we establish an Itô formula. In the last section, we show how the Stratonovith integral is related to a Skorohod-like integral and how a Itô-like process constructed from such an integral is modified through an absolutely continuous change of probability.

2 Preliminaries

This section is only devoted to the presentation of the tools of deterministic fractional calculus we shall use in the sequel. For $f \in \mathcal{L}^1([0,1];\, dt)$, (denoted by $\mathcal{L}^1$ for short) the left and right fractional integrals of f are defined by :

$$(I_{0^+}^{\gamma} f)(x) \stackrel{def}{=} \frac{1}{\Gamma(\gamma)} \int_0^x f(t)(x-t)^{\gamma-1} dt \ , \ x \geq 0,$$
$$(I_{1^-}^{\gamma} f)(x) \stackrel{def}{=} \frac{1}{\Gamma(\gamma)} \int_x^1 f(t)(t-x)^{\gamma-1} dt \ , \ x \leq 1,$$

where $\gamma > 0$ and $I^0_{0^+} = I^0_{1^-} = \mathrm{Id}$. For any $\gamma \geq 0$, any $f \in \mathcal{L}^p$ and $g \in \mathcal{L}^q$ where $p^{-1} + q^{-1} \leq \gamma$, we have :

$$\int_0^1 f(s)(I^\gamma_{0^+} g)(s)\, ds = \int_0^1 (I^\gamma_{1^-} f)(s) g(s)\, ds. \tag{2}$$

The Besov-Liouville space $I^\gamma_{0^+}(\mathcal{L}^p) \overset{not}{=} I^+_{\gamma,p}$ is usually equipped with the norm :

$$\|I^\gamma_{0^+} f\|_{I^+_{\gamma,p}} = \|f\|_{\mathcal{L}^p}. \tag{3}$$

Analogously, the Besov-Liouville space $I^\gamma_{1^-}(\mathcal{L}^p) \overset{not}{=} I^-_{\gamma,p}$ is usually equipped with the norm :

$$\|I^{-\gamma}_{1^-} f\|_{I^-_{\gamma,p}} = \|f\|_{\mathcal{L}^p}.$$

We then have the following continuity results (see [FdLP99, SKM93]) :

Proposition 2.1. *i. If $0 < \gamma < 1$, $1 < p < 1/\gamma$, then $I^\gamma_{0^+}$ is a bounded operator from $\mathcal{L}^p$ into $\mathcal{L}^q$ with $q = p(1-\gamma p)^{-1}$.*

ii. For any $0 < \gamma < 1$ and any $p \geq 1$, $I^+_{\gamma,p}$ is continuously embedded in $\mathrm{Hol}(\gamma - 1/p)$ provided that $\gamma - 1/p > 0$. $\mathrm{Hol}(\nu)$ denotes the space of Hölder-continuous functions, null at time 0, equipped with the usual norm.

iii. For any $0 < \gamma < \beta < 1$, $\mathrm{Hol}(\beta)$ is compactly embedded in $I_{\gamma,\infty}$.

iv. By $I^{-\gamma}_{0^+}$, respectively $I^{-\gamma}_{1^-}$, we mean the inverse map of $I^\gamma_{0^+}$, respectively $I^\gamma_{1^-}$. The relation $I^\gamma_{0^+} I^\beta_{0^+} f = I^{\gamma+\beta}_{0^+} f$ holds whenever $\beta > 0$, $\gamma + \beta > 0$ and $f \in \mathcal{L}^1$.

v. For $\gamma p > 1$, the spaces $I^+_{\gamma,p}$ and $I^-_{\gamma,p}$ are canonically isomorphic. We will thus use the notation $I_{\gamma,p}$ to denote any of this spaces. This property isn't any more true for $\gamma p > 1$, see Lemma 2.3 and text below Definition 4.1.

We now define the Besov-Liouville spaces of negative order and show that they are in duality with Besov-Liouville of positive order (it is likely that this exists elsewhere in the literature but we have not found any reference so far). Denote by $\mathcal{D}_+$ the space of $\mathcal{C}^\infty$ functions defined on $[0,1]$ and such that $\phi^{(k)}(0) = 0$. Analogously, set $\mathcal{D}_-$ the space of $\mathcal{C}^\infty$ functions defined on $[0,1]$ and such that $\phi^{(k)}(1) = 0$. They are both equipped with the projective topology induced by the semi-norms $p_k(\phi) = \sum_{j \leq k} \|\phi^{(j)}\|_\infty$. Let $\mathcal{D}'_+$, resp. $\mathcal{D}'_-$, be their strong topological dual. It is straightforward that $\mathcal{D}_+$ is stable by $I^\gamma_{0^+}$ and $\mathcal{D}_-$ is stable $I^\gamma_{1^-}$, for any $\gamma \in \mathbf{R}$. Hence, guided by (2), we can define the fractional integral of any distribution (i.e., an element of $\mathcal{D}'_-$ or $\mathcal{D}'_+$):

$$\text{For } T \in \mathcal{D}'_-;\ I^{\gamma}_{0^+}T : \ \phi \in \mathcal{D}_- \mapsto < T, I^{\gamma}_{1^-}\phi >_{\mathcal{D}'_-,\mathcal{D}_-},$$
$$\text{For } T \in \mathcal{D}'_+;\ I^{\gamma}_{1^-}T : \ \phi \in \mathcal{D}_+ \mapsto < T, I^{\gamma}_{0^+}\phi >_{\mathcal{D}'_+,\mathcal{D}_+}.$$

We introduce now our Besov spaces of negative order by

Definition 2.1. *For $\gamma > 0$ and $r > 1$, $I^+_{-\gamma,r}$ (resp. $I^-_{-\gamma,r}$) is the space of distributions such that $I^{\gamma}_{0^+}T$ (resp. $I^{\gamma}_{1^-}T$) belongs to $\mathcal{L}^r$. The norm of an element T in this space is the norm of $I^{\gamma}_{0^+}T$ in $\mathcal{L}^r$ (resp. of $I^{\gamma}_{1^-}T$).*

Theorem 2.1. *For $\gamma > 0$ and $r > 1$, the dual space of $I^+_{\gamma,r}$ (resp. $I^-_{\gamma,r}$) is canonically isometrically isomorphic to $I^{-\gamma}_{1^-}(\mathcal{L}^{r^*})$ (resp. $I^{-\gamma}_{0^+}(\mathcal{L}^{r^*})$,) where $r^* = r(r-1)^{-1}$.*

Proof. Let T be in $\mathcal{D}'_+$, we have:

$$\begin{aligned}\sup_{\phi:\|\phi\|_{I^+_{\gamma,r}}=1} | < T, \phi > | &= \sup_{\psi:\|\psi\|_{\mathcal{L}^r}=1} | < T, I^{\gamma}_{0^+}\phi > | \\ &= \sup_{\psi:\|\psi\|_{\mathcal{L}^r}=1} | < I^{\gamma}_{1^-}T, \phi > |\end{aligned}$$

hence by the Hahn-Banach theorem,

$$T \in (I^+_{\gamma,r})' \Longleftrightarrow \sup_{\phi:\|\phi\|_{I^+_{\gamma,r}}=1} | < T, \phi > | < \infty \Longleftrightarrow I^{\gamma}_{1^-}T \in \mathcal{L}^{r^*},$$

and $\|T\|_{(I^+_{\gamma,r})'} = \|T\|_{I^{-\gamma}_{1^-}(\mathcal{L}^{r^*})}$. The same reasoning also holds for $(I^-_{\gamma,r})'$. $\square$

Theorem 2.2. *For $\beta \geq \gamma \geq 0$ and $r > 1$, $I^{\beta}_{1^-}$ is continuous from $I^-_{-\gamma,r}$ into $I^-_{\beta-\gamma,r}$.*

Proof. Since T belongs to $I^-_{-\gamma,r} = (I_{\gamma,r^*})'$, we have:

$$| < I^{\beta}_{1^-}T, \phi > | = | < T, I^{\beta}_{0^+}\phi > | \leq c\|I^{\beta}_{0^+}\phi\|_{I_{\gamma,r^*}} = c\|I^{\beta-\gamma}_{0^+}\phi\|_{\mathcal{L}^{r^*}}.$$

Thus, $I^{\beta}_{1^-}T$ is a continuous linear form on $I^+_{\gamma-\beta,r^*}$ and thus belongs to the dual of this space which, according to the previous theorem, is exactly $I^-_{\beta-\gamma,r}$. $\square$

For $\eta > 0$ and $p \in [1, +\infty)$, the Slobodetzki space $\mathcal{S}_{\eta,p}$ is the closure of $\mathcal{C}^1$ functions with respect to the semi-norm:

$$\|f\|^p_{\mathcal{S}_{\eta,p}} = \iint_{[0,1]^2} \frac{|f(x) - f(y)|^p}{|x - y|^{1+p\eta}}\, dx\, dy,$$

For $\eta = 0$, we simply have $\mathcal{S}_{0,p} = L^p([0,1])$. We then have the following continuity results (see [FdLP99, Zÿ98]) :

Proposition 2.2. *i. For any $0 < \gamma < 1$ and any $p \geq 1$, $\mathcal{S}_{\gamma,p}$ is continuously embedded in* $\mathrm{Hol}(\gamma - 1/p)$ *provided that $\gamma - 1/p > 0$.* $\mathrm{Hol}(\nu)$ *denotes the space of Hölder-continuous functions, null at time 0, equipped with the usual norm.*

For $0 < \gamma < 1/p$, $\mathcal{S}_{\gamma,p}$ is compactly embedded in $L^{p(1-\gamma p)^{-1}}([0,1])$. Moreover, if $p = 2$, the embedding of $\mathcal{S}_{\gamma,p}$ into $L^2([0,1])$ is Hilbert-Schmidt.

ii. It is proved in [FdLP99] that for $1 \geq a > b > c > 0$ that we the following embeddings are continuous (even compact)

$$\mathcal{S}_{a,p} \subset I^+_{b,p} \subset \mathcal{S}_{c,p}. \tag{4}$$

iii. For any $0 < \gamma < \beta < 1$, $\mathrm{Hol}(\beta)$ *is compactly embedded in $\mathcal{S}_{\gamma,\infty}$.*

iv. Let $a > 0$, $1 < p \leq q < \infty$. Suppose $b = a - 1/p + 1/q > 0$. Then $\mathcal{S}_{a,p}$ is continuously embedded in $\mathcal{S}_{b,q}$, see [Ada75].

One of the key property we shall use, is this result due to Tambaca [Tam01].

Lemma 2.1. *Let $r, s \in [0, 1/2)$ and let $t = r + s - 1/2 \geq 0$. For $f \in \mathcal{S}_{s,2}$, $g \in \mathcal{S}_{r,2}$, the product fg belongs to $\mathcal{S}_{t,2}$ and we have:*

$$\|fg\|_{\mathcal{S}_{t,2}} \leq c\|f\|_{\mathcal{S}_{r,2}}\|g\|_{\mathcal{S}_{s,2}}.$$

From this Lemma and the embeddings of Eqn. (4), we have:

Corollary 2.1. *Let $r, s \in [-\infty, 1/2)$ and let $t < r + s - 1/2$. For $f \in I_{s,2}$, $g \in I_{r,2}$, the product fg belongs to $I_{t,2}$ and we have:*

$$\|fg\|_{I_{t,2}} \leq c\|f\|_{I_{r,2}}\|g\|_{I_{s,2}}.$$

We will need a similar result in the simpler situation where r is greater than $1/2$.

Lemma 2.2. *Let $r > 1/2$, for f and g in $I_{r,2}$, we have*

$$\|fg\|_{\mathcal{S}_{r,2}} \leq c\|f\|_{\mathcal{S}_{r,2}}\|g\|_{\mathcal{S}_{r,2}}. \tag{5}$$

Proof. Since $r > 1/2$, f and g are continuous and $\|f\|_\infty \leq c\|f\|_{\mathcal{S}_{r,2}}$. The same holds for g. Thus,

$$\begin{aligned}\|fg\|^2_{\mathcal{S}_{r,2}} &\leq \iint_{[0,1]^2} \left(\frac{|f(x)|^2(g(x)-g(y))^2}{|x-y|^{1+2r}} + \frac{|g(y)|^2(f(x)-f(y))^2}{|x-y|^{1+2r}} \right) \mathrm{d}x\, \mathrm{d}y \\ &\leq c\Big(\|f\|^2_\infty\|g\|^2_{\mathcal{S}_{r,2}} + \|g\|^2_\infty\|f\|^2_{\mathcal{S}_{r,2}}\Big),\end{aligned}$$

and the result follows. $\square$

One could probably work with only one family of spaces (i.e., either $I_{\alpha,p}$ or $\mathcal{S}_{\alpha,p}$) but depending on the properties, some are easier to verify in the setting of Riemann-Liouville spaces and some in the setting of Slobodetzki spaces, see for instance the property below.

Lemma 2.3. *Let* $\gamma > \tilde{\gamma} > 1/2$ *and* $f \in \mathcal{S}_{\gamma,2}$ *then* $(f - f(t))\mathbf{1}_{[0,t]}$ *belongs to* $\mathcal{S}_{\tilde{\gamma},2}$.

Proof. First note that f is $(\gamma - 1/2)$-Hölder continuous thus that $f - f(t)$ is well defined. Moreover,

$$\begin{aligned}\iint_{[0,1]^2} & \frac{|(f(x) - f(t))\mathbf{1}_{[0,t]} - (f(y) - f(t))\mathbf{1}_{[0,t]}|^2}{|x-y|^{1+2\tilde{\gamma}}}\, dx\, dy \\ &= \iint_{[0,t]^2} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2\tilde{\gamma}}}\, dx\, dy + 2\iint_{[0,t]\times[0,1]} \frac{|f(x) - f(t)|^2}{|x-y|^{1+2\tilde{\gamma}}}\, dx\, dy \\ &\qquad \le \|f\|^2_{\tilde{\gamma},2}\Big(1 + 2\iint_{[0,t]\times[t,1]} \frac{|x-t|^{2\gamma-1}}{|x-y|^{1+2\tilde{\gamma}}}\, dx\, dy\Big) \le c\|f\|^2_{\tilde{\gamma},2}.\end{aligned}$$

□

3 Volterra processes

Consider that we are given a deterministic Hilbert-Schmidt linear map, K, satisfying:

Hypothesis I. There exists $\alpha > 0$ such that K is continuous, one-to-one, from $\mathcal{L}^2([0,1])$ into $I_{\alpha+1/2,2}$. Moreover, K is triangular, i.e., for any $\lambda \in [0,1]$, the set $\mathcal{N}_\lambda = \{f : \ f(t) = 0 \text{ for } t \le \lambda\}$ is invariant by K.

Remark 3.1. Since K is Hilbert-Schmidt from $\mathcal{L}^2([0,1])$ into itself, there exists a measurable kernel $K(.,.)$ such that

$$Kf(t) = \int_0^1 K(t,s) f(s)\, \mathrm{d}s.$$

The triangularity of K is equivalent to $K(t,s) = 0$ for $s > t$, i.e.,

$$Kf(t) = \int_0^t K(t,s) f(s)\, \mathrm{d}s.$$

Consider now the kernel $R(t,s)$ defined by

$$R(t,s) := \int_0^{t\wedge s} K(t,r)K(s,r)\, \mathrm{d}r.$$

The map associated to R, i.e., $Rf(t) = \int_0^1 R(t,s)f(s)\,\mathrm{d}s$, is equal to KK^* and for any $\beta_1,\dots,\beta_n$ any $t_1,\dots,t_n$, we have

$$\sum_{i,j}\beta_i\beta_j R(t_i,t_j) = \int K^*(\sum\beta_j\varepsilon_{t_j})(s)^2\,\mathrm{d}s \geq 0,$$

so that $R(t,s)$ is a positive kernel and we can speak of the centered Gaussian process of covariance kernel R. Let X be this process and be the subject of our study.

Lemma 3.1. *The process X has a modification with a.s. continuous sample-paths.*

Proof. We have

$$\begin{aligned}\mathrm{E}\left[(X_t-X_s)^2\right] &= \int_0^t K(t,r)^2\,\mathrm{d}r + \int_0^s K(s,r)^2\,\mathrm{d}r - 2\int_0^{t\wedge s} K(t,r)K(s,r)\,\mathrm{d}r\\ &= K(K(t,.)-K(s,.))(t) - K(K(t,.)-K(s,.))(s)\\ &\leq c|t-s|^\alpha \Big(\int_0^1 (K(t,r)-K(s,r))^2\,\mathrm{d}r\Big)^{1/2}.\end{aligned}$$

Expanding the square in the last integral, we get the right hand side of the first equation, thus

$$\mathrm{E}\left[(X_t-X_s)^2\right]^{1/2} \leq c|t-s|^\alpha.$$

Kolmogorov Lemma entails that X has a modification with Hölder continuous sample paths of any order less than α. □

We thus now work on the Wiener space $\Omega = \mathcal{C}_0([0,1];\mathbf{R})$, the Cameron-Martin space is $H = K(\mathcal{L}^2([0,1]))$ and P, the probability on Ω under which the canonical process, denoted by X, is a centered Gaussian process of covariance kernel R. The norm of $h = K(g)$ in H is the norm of g in $\mathcal{L}^2([0,1])$.

A mapping ϕ from Ω into some separable Hilbert space X is called cylindrical if it is of the form $\phi(w) = \sum_{i=1}^d f_i(\langle v_{i,1},w\rangle,\cdots,\langle v_{i,n},w\rangle)x_i$ where for each i, $f_i \in \mathcal{C}_0^\infty(\mathbf{R}^n,\mathbf{R})$ and $(v_{i,j},\ j=1\dots n)$ is a sequence of Ω^* such that $(\tilde{v}_{i,j},\ j=1\dots n)$ (where $\tilde{v}_{i,j}$ is the image of $v_{i,j}$ under the injection $\Omega^\star \hookrightarrow \mathcal{L}^2([0,1])$) is an orthonormal system of $\mathcal{L}^2([0,1])$. For such a function we define $\nabla\phi$ as

$$\nabla\phi(w) = \sum_{i,j=1} \partial_j f_i(\langle v_{i,1},w\rangle,\cdots,\langle v_{i,n},w\rangle)\tilde{v}_{i,j}\otimes x_i.$$

From the quasi-invariance of the Wiener measure [Ü95], it follows that ∇ is a closable operator on $L^p(\Omega;X)$, $p\geq 1$, and we will denote its closure with the same notation. The powers of ∇ are defined by iterating this procedure. For $p>1$, $k\in\mathbb{N}$, we denote by $\mathbb{D}_{p,k}(X)$ the completion of X-valued cylindrical functions under the following norm

$$\|\phi\|_{p,k} = \sum_{i=0}^{k} \|\nabla^i \phi\|_{L^p(\Omega; X \otimes \mathcal{L}^2([0,1])^{\otimes i})} \,.$$

Remark 3.2. Note that the Sobolev spaces $\mathcal{S}_{\alpha,p}$ enjoy the useful property of p-admissibility (after [FdLP91]) and thus for any $0 < \gamma < 1$ and any $p \geq 1$, the spaces $\mathbb{D}_{p,k}(\mathcal{S}_{\alpha,p})$ and $\mathcal{S}_{\alpha,p}(\mathbb{D}_{p,k})$ are isomorphic.

The divergence, denoted δ is the adjoint of ∇: v belongs to $\operatorname{Dom}_p \delta$ whenever for any cylindrical ϕ,

$$|\mathrm{E}\left[\int_0^1 u_s \nabla_s \phi \, \mathrm{d}s\right]| \leq c\|\phi\|_{L^p}$$

and for such a process v,

$$\mathrm{E}\left[\int_0^1 u_s \nabla_s \phi \, \mathrm{d}s\right] = \mathrm{E}\left[\phi \, \delta u\right].$$

It is easy to show (see [DÜ99]) that $\{B_t := \delta(\mathbf{1}_{[0,t]}),\, t \geq 0\}$ is a standard Brownian motion such that $\delta u = \int u_s \, \mathrm{d}B_s$ for any square integrable adapted processes u and which satisfies

$$X_t = \int_0^t K(t,s) \, \mathrm{d}B_s.$$

Moreover, B and X have the same filtration. In view of the last identity and because K is lower triangular, we decided to name such a process, a Gaussian Volterra process. The analysis of processes of the same kind where B is replaced by a jump processes is the subject of our current investigations with N. Savy.

Example 1. The first example is the so-called Lévy fractional Brownian motion of Hurst index H, defined as

$$\frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \, \mathrm{d}B_s.$$

This amounts to say that $K = I_{0^+}^{H+1/2}$, thus that hypothesis I and II are immediately satisfied, with $\alpha = H$, in view of the semi-group properties of fractional integration.

Example 2. The other classical example is the fractional Brownian motion with stationary increments of Hurst index H, for which

$$K(t,s) = K_H(t,r) := \frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})} F(\frac{1}{2} - H, H - \frac{1}{2}, H + \frac{1}{2}, 1 - \frac{t}{r}) 1_{[0,t)}(r). \qquad (6)$$

The Gauss hyper-geometric function $F(\alpha,\beta,\gamma,z)$ (see [NU88]) is the analytic continuation on $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\backslash\{-1,-2,\dots\}\times\{z\in\mathbb{C}, Arg|1-z|<\pi\}$ of the power series

$$\sum_{k=0}^{+\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}z^k,$$

and

$$(a)_0 = 1 \text{ and } (a)_k \overset{def}{=} \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\dots(a+k-1).$$

We know from [SKM93] that K_H is an isomorphism from $\mathcal{L}^2([0,1])$ onto $I^+_{H+1/2,2}$ and

$$K_H f = I^{2H}_{0^+} x^{1/2-H} I^{1/2-H}_{0^+} x^{H-1/2} f \text{ for } H \le 1/2,$$
$$K_H f = I^{1}_{0^+} x^{H-1/2} I^{H-1/2}_{0^+} x^{1/2-H} f \text{ for } H \ge 1/2.$$

It follows easily that Hypothesis I and II are satisfied with $\alpha = H$.

Example 3. Beyond these two well known cases, we can investigate the case of $K(t,s) = K_{H(t)}(t,s)$ for a deterministic function H. This is the process studied in [BBCI99]. It seems interesting to analyze since statistical investigations via wavelets have shown that the local Hölder exponent of some real signals (in telecommunications) is varying with time and this situation can't be reflected with a model based on fBm since its Hölder regularity is everywhere equal to its Hurst index.

Lemma 3.2. *For $f \in \mathcal{L}^2$, for $H_1 > H_2 \ge \gamma > 0$, we have*

$$|K_{H_2}f(s) - K_{H_2}f(t)| \le c|t-s|^{\gamma}\|f\|_{\mathcal{L}^2}, \tag{7}$$
$$|K_{H_1}f(s) - K_{H_2}f(s)| \le c|H_1-H_2|\|f\|_{\mathcal{L}^2}, \tag{8}$$

where c is a constant independent of H_1, H_2 and f.

Proof. Since H_2 is greater than γ, $K_{H_2}f$ belongs to $I_{\gamma+1/2,2}$, and (7) follows directly from the embedding of $I_{\gamma+1/2,2}$ into $\mathrm{Hol}(\gamma)$.

Another expression of the hypergeometric function is given by:

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{c-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t.$$

Classical and tedious computations show that for $H \in [h_1+\varepsilon, h_2-\varepsilon]$,

$$|\frac{d}{dH}K_H(t,s)| \le c_\varepsilon \sup_{H\in(H_1,H_2)} |K_H(t,s)|,$$

where $c_\varepsilon = \sup_{t\in[0,1]} |t^\varepsilon \ln t|$. It thus entails that

$$|K_{H_2}(t,s) - K_{H_1}(t,s)| \le c_\varepsilon \sup_{H\in(H_1,H_2)} |K_H(t,s)||H_2 - H_1|.$$

Cauchy-Schwarz inequality yields to (8). □

Theorem 3.1. *Let H belong to $\mathcal{S}_{1/2+\alpha,2}$ and be such that* $\inf_t H(t) > 1/2$, *then* $K(t,s) = K_{H(t)}(t,s)$ *satisfies I for any* $\alpha < \inf_t H(t) - 1/2$.

Proof. Let f belong to $\mathcal{L}^2$, set $\gamma = \inf_t H(t)$ and let $\alpha < \gamma - 1/2$. According to the previous lemma, we have

$$\begin{aligned}
\|Kf\|^2_{\mathcal{S}_{1/2+\alpha,2}} &= \iint_{[0,1]^2} \frac{|K_{H(t)} f(t) - K_{H(s)} f(s)|^2}{|t-s|^{2+2\alpha}} \,\mathrm{d}t\,\mathrm{d}s \\
&\le 2 \iint_{[0,1]^2} \frac{|K_{H(t)} f(t) - K_{H(t)} f(s)|^2}{|t-s|^{2+2\alpha}} \,\mathrm{d}t\,\mathrm{d}s \\
&\qquad + 2 \iint_{[0,1]^2} \frac{|K_{H(t)} f(s) - K_{H(s)} f(s)|^2}{|t-s|^{2+2\alpha}} \,\mathrm{d}t\,\mathrm{d}s \\
&\le c\,\| f\|^2_{\mathcal{L}^2} \iint_{[0,1]^2} \frac{|t-s|^{2\gamma}}{|t-s|^{1+2\alpha}} \,\mathrm{d}t\,\mathrm{d}s + c\,\| f\|^2_{\mathcal{L}^2} \iint_{[0,1]^2} \frac{|H(t)-H(s)|^2}{|t-s|^{2+2\alpha}} \,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

The right-hand-side is finite by hypothesis and thus K is continuous from $\mathcal{L}^2$ into $\mathcal{S}_{1/2+\alpha,2}$. □

4 Stratonovitch integral

Starting from scratch and trying to define a stochastic integral with respect to X by a limit of a sequence of finite sums, we have two main choices: Either we discretize X (or more probably dX) or we discretize B (likely dB) and then derive a discretization of dX. The first approach yields two possibilities: for a partition π whose points are denoted by $0 = t_0 < t_1 < \ldots < t_n = T$, we can consider

$$\mathrm{RS}_\pi(u) = \sum_{t_i\in\pi} u(t_i)\Delta X_i \text{ or} \tag{9}$$

$$\mathrm{SS}_\pi(u) = \sum_{t_i\in\pi} \frac{1}{\delta_i}\Big(\int_{t_i}^{t_{i+1}} u(s)\,\mathrm{d}s\Big)\,\Delta X_i, \tag{10}$$

where $\delta_i = t_{i+1} - t_i$ and $\Delta X_i = X(t_{i+1}) - X(t_i)$. They are both reminiscences of respectively Riemann and Skorohod-Stratonovitch sums as defined in [Nua95].

In the other approach, we first linearize B and then look at the approximation of X it yields to. Let

$$B^{\pi}(t) = B(t_i) + \frac{1}{\delta_i}\Delta B_i(t-t_i) \text{ for } t \in [t_i, t_{i+1}),$$

and

$$\begin{aligned} X^{\pi}(t) &= \sum_{t_i\in\pi} \frac{1}{\delta_i}\int_{t_i}^{t_{i+1}} K(t,s)\, ds\, \Delta B_i \\ &= \sum_{t_i\in\pi} \frac{1}{\delta_i} K(\mathbf{1}_{[t_i,t_{i+1}]})(t)\Delta B_i. \end{aligned}$$

It follows that it is reasonable to consider

$$R_T^{\pi}(u) := \sum_{t_i\in\pi} \frac{1}{\delta_i}\left\{\int_0^T u(t)\frac{d}{dt}K(\mathbf{1}_{[t_i,t_{i+1}]})(t)\,\mathrm{d}t\right\}\,\Delta B_i,$$

under the additional hypothesis that for any $b>0$, the function $K(\mathbf{1}_{[0,b]})$ is differentiable with a square integrable derivative. For u sufficiently smooth in the sense of the calculus of variations, we have

$$\begin{aligned} R_T^{\pi}(u) = \delta\Big(\sum_{t_i\in\pi} \frac{1}{\delta_i}\int_0^T u(t)\frac{d}{dt}K(\mathbf{1}_{[t_i,t_{i+1}]})(t)\, dt\,\mathbf{1}_{[t_i,t_{i+1}]}\Big) \\ + \sum_{t_i\in\pi} \frac{1}{\delta_i}\int_{t_i}^{t_{i+1}}\int_0^T \nabla_r u(t)\frac{d}{dt}K(\mathbf{1}_{[t_i,t_{i+1}]})(t)\,\mathrm{d}t\,\mathrm{d}r. \end{aligned}$$

Using $\mathcal{K}_T^*$, the formal adjoint of $\mathcal{K} := I_{0^+}^{-1}\circ K$ on $\mathcal{L}^2([0,T])$, we have

$$R_T^{\pi}(u) \quad = \quad \delta\Big(\sum_{t_i\in\pi}\frac{1}{\delta_i}\int_{t_i}^{t_{i+1}}\mathcal{K}_T^* u(t)\,\mathrm{d}t\Big) \quad + \quad \sum_{t_i\in\pi}\frac{1}{\delta_i}\iint\limits_{[t_i,t_{i+1}]^2}\mathcal{K}_T^*(\nabla_r u)(t)\,\mathrm{d}t\,\mathrm{d}r. \quad (11)$$

We now recognize the Skorohod-Stratonovitch sum associated to the standard Brownian motion B and to the integrand $\mathcal{K}_T^* u$. For the sequel to be meaningful, we need to assume that the map $\mathcal{K}$ exists. This is guaranteed for $\alpha \geq 1/2$, since $I^+_{\alpha+1/2,2}$ is embedded in the set of absolutely continuous functions with square integrable derivative, but for $\alpha < 1/2$, we need to introduce an additional hypothesis.

Hypothesis II. We assume that for any $T\in[0,1]$, the map $\mathcal{K} = I_{0^+}^{-1}\circ K$ is a densely defined, closable operator from $\mathcal{L}^2([0,T])$ into itself and that its domain contains a dense subset, $\mathcal{D}$, stable by the maps p_T, for any $T\in[0,1]$, where $p_T f \equiv f\mathbf{1}_{[0,T)}$. We denote by $\mathcal{K}_T^*$ its adjoint in $\mathcal{L}^2([0,T])$. We assume furthermore that $\mathcal{K}_1^*$ is continuous from $I^{1^-}_{1/2-\alpha,p}$ into $\mathcal{L}^2([0,T])$, for any $p\geq 2$.

Remark 4.1. In the preceding examples, $\mathcal{D}$ may be taken to $I_{(1/2-\alpha)^+,2}$.

Remark 4.2. For the sake of simplicity, we will speak of the domains of $\mathcal{K}$ and $\mathcal{K}_T^*$ independently of the position of α with respect to $1/2$. It must be plain that for $\alpha > 1/2$, $\operatorname{Dom}\mathcal{K} = \mathcal{L}^2([0,1])$ and $\operatorname{Dom}\mathcal{K}_T^* = \mathcal{L}^2([0,T])$.

Remark 4.3. Since $I_{1^-}^1(\varepsilon_t) = \mathbf{1}_{[0,t]}$, we have

$$\mathcal{K}^*(\mathbf{1}_{[0,t]}) = K^*(\varepsilon_t) = K(t,.).$$

This means that $\mathcal{K}_t^*$ is identical to the operator denoted by $I_t^{K_H}$ in [CCM02].

Notation 1. For any $p \geq 1$, we denote by p^* the conjugate of p. For any linear map A, we denote by A_T^*, its adjoint in $\mathcal{L}^2([0,T])$. We denote by c any irrelevant constant appearing in the computations, c may vary from one line to another.

Definition 4.1. *Assume that Hypothesis I holds for* $\alpha \geq 1/2$. *We say that u is Stratonovitch integrable on* $[0,T]$ *whenever the family* $R_T^\pi(u)$, *defined in (11), converges in probability as* $|\pi|$ *goes to* 0. *In this case the limit will be denoted by* $\int_0^T u_s \circ\, dX_s$.

This definition could be theoretically extended to $\alpha < 1/2$ but would be practically unusable. Indeed, as we shall see below, when $\alpha < 1/2$, the convergence of the second sum of $R_T^\pi(u)$ requires that u belongs to $I_{1+\eta-\alpha,2}$ for some $\eta > 0$ and $\mathcal{K}_T^*$ to be continuous from this space to a space of Holderian functions. Since $1+\eta-\alpha-1/2 > 0$, the two spaces $I_{0^+}^{1+\eta-\alpha}(\mathcal{L}^2([0,T]))$ and $I_{T^-}^{1+\eta-\alpha}(\mathcal{L}^2([0,T]))$ are not canonically isomorphic (if u belongs to the first one then $u(0) = 0$ whereas when u belongs to the latter, $u(T) = 0$). We thus have to specify to which one u belongs exactly. In view of the example of the Lévy fractional Brownian where $\mathcal{K}_T^* = I_{T^-}^{H-1/2}$, it is more convenient to assume that u belongs to $I_{T^-}^{1+\eta-H}(\mathcal{L}^2([0,T]))$ and thus that $u(T)$ is equal to 0. That raises a problem because the restriction of an element of $I_{T^-}^{1+\eta-H}(\mathcal{L}^2([0,T]))$ to a shorter interval, say $[0,S]$, does not belong $I_{S^-}^{1+\eta-H}(\mathcal{L}^2([0,S]))$ so that, we can't see $\int_0^S u(r) \circ \mathrm{d}X_r$ as $\int_0^T u(r)\mathbf{1}_{[0,S]}(r) \circ \mathrm{d}X_r$.

On the other hand, since $(u-u(S))\mathbf{1}_{[0,S]}$ belongs to $I_{S^-}^{1+\eta-H}(\mathcal{L}^2([0,S]))$ as soon as u belongs to $I_{T^-}^{1+\eta-H}(\mathcal{L}^2([0,T]))$, it is reasonable to consider $R_T^\pi(u-u(T))$. For the limit to stay the same, we have to add the term $u(T)X(T)$. Indeed, the well known relationship (see [Nua95, Ü95])

$$\delta(a\xi) = a\delta\xi - \int_0^1 \nabla_r a\xi(r)\,\mathrm{d}r, \tag{12}$$

for $a \in \mathbb{D}_{2,1}$ and $\xi \in \mathcal{L}^2(\Omega \times [0,1])$, entails that

$$R_T^\pi(u) = R_T^\pi(u-u(T)) + u(T)X^\pi(T). \tag{13}$$

As a conclusion, for $\alpha < 1/2$, the definitive definition is

Definition 4.2 (Definition for $\alpha < 1/2$). *Assume that Hypothesis I and II hold for $\alpha < 1/2$. We say that u is Stratonovitch integrable on $[0,T]$, whenever the family $R_T^\pi(u-u(T))$ converges in probability as $|\pi|$ goes to 0. In this case, we set*

$$\int_0^T u_s \circ dX_s = \lim_{|\pi|\to 0} R_T^\pi(u-u(T)) + u(T)X(T). \tag{14}$$

In view of the preceding discussion, the following lemma will play a key role in the sequel.

Lemma 4.1. *For $T \in (0,1]$, let $p_T f$ denote the restriction of f to $[0,T)$. For any $f \in \operatorname{Dom} \mathcal{K}_1^*$, f belongs to $\operatorname{Dom} \mathcal{K}_T^*$, $p_T f$ belongs to $\operatorname{Dom} \mathcal{K}_1^*$ and we have*

$$p_T \mathcal{K}_1^*(p_T f) \equiv \mathcal{K}_T^*(f). \tag{15}$$

Proof. Since K is triangular, for $g \in \mathcal{D}$, $p_T g$ belongs to $\operatorname{Dom} \mathcal{K}$ and $p_T K g = p_T K(p_T g) = K p_T g$. By derivation, it follows that $p_T \mathcal{K} g = p_T \mathcal{K} p_T g = \mathcal{K} p_T g$, so that, for $f \in \operatorname{Dom} \mathcal{K}_1^*$,

$$\begin{aligned}
|\int_0^t f(s)\mathcal{K}g(s)\,\mathrm{d}s| &= |\int_0^1 (p_T f)(s)\mathcal{K}g(s)\,\mathrm{d}s| \\
&= |\int_0^1 f(s)(p_T\mathcal{K}g)(s)\,\mathrm{d}s| \\
&= |\int_0^1 f(s)\mathcal{K}(p_T g)(s)\,\mathrm{d}s| \\
&\le c\|p_T g\|_{\mathcal{L}^2([0,1])} = c\|g\|_{\mathcal{L}^2([0,T])}.
\end{aligned}$$

By density, this identity remains true for $g \in \operatorname{Dom} \mathcal{K}$, thus this means that f belongs to $\operatorname{Dom} \mathcal{K}_T^*$ and that $p_T f$ belongs to $\operatorname{Dom} \mathcal{K}_1^*$.

For $g \in \mathcal{L}^2([0,T]) \cap \operatorname{Dom} \mathcal{K}$, we denote by $\tilde{g}$ its extension to $\mathcal{L}^2([0,T])$ defined by $\tilde{g}(s) = 0$ whenever $s \ge T$. We have

$$\begin{aligned}
\int_0^T p_T \mathcal{K}_1^* p_T f(s) g(s)\,\mathrm{d}s &= \int_0^1 \mathcal{K}_1^* p_T f(s) p_T \tilde{g}(s)\,\mathrm{d}s \\
&= \int_0^1 p_T f(s)\mathcal{K}(p_T\tilde{g})(s)\,\mathrm{d}s \\
&= \int_0^T f(s)\mathcal{K}g(s)\,\mathrm{d}s \\
&= \int_0^T \mathcal{K}_T^* f(s) g(s)\,\mathrm{d}s,
\end{aligned}$$

where the last equality follows by the first part of the proof and the definition of the adjoint of a linear map. Since g can be arbitrary, (15) follows by identification. □

Theorem 4.1. *Let $\alpha < 1/2$ and $p \geq 2$. Assume that Hypothesis I and II hold. Assume furthermore that there exists $\sigma > 1/p$ and $\eta > 0$, such that $\mathcal{K}_1^*$ is continuous from $I^{1^-}_{\sigma,p}$ into $\mathrm{Hol}(\eta)$. If u belongs to $\mathbb{D}_{p,1}(I^{1^-}_{\sigma+\varepsilon,p})$, for some $\varepsilon > 0$, then for any $T \in [0,1]$, there exists a measurable and integrable process, denoted by $\tilde{D}_T u$ such that, for any s, any $0 \leq a < b < 1$,*

$$E\left[\int_a^b |\mathcal{K}_T^*(\nabla_r(u-u(T)))(s) - \tilde{D}_T u(r)|^p \, dr\right] \\ \leq c\, E\left[\int_0^1 |s-r|^{p\eta} \|\nabla_r u\|^p_{I^{1^-}_{\sigma+\varepsilon,p}} \, dr\right]. \quad (16)$$

Moreover,

$$E\left[\|\int_0^{\cdot} \tilde{D}_T u(r)\, dr\|^p_{I^+_{1,p}}\right] \leq c\|u\|^p_{\mathbb{D}_{p,1}(I^{1^-}_{\sigma+\varepsilon,p})}. \quad (17)$$

Proof. Since $\sigma > 1/p$, u is continuous and we can speak unambiguously of $u(T)$. The assumed continuity of $\mathcal{K}_1^*$ entails that $\mathcal{K}_T^*(u-u(T))$ belongs to $\mathbb{D}_{p,1}(\mathrm{Hol}(\eta))$ and that

$$\mathrm{E}\left[\int_a^b |\nabla_r \mathcal{K}_T^*(u-u(T))(s) - \nabla_r \mathcal{K}_T^*(u-u(T))(\tau)|^p \, \mathrm{d}r\right] \\ \leq c\, \mathrm{E}\left[\int_0^1 |s-\tau|^{p\eta} \|\nabla_r u\|^p_{I^{1^-}_{\sigma+\varepsilon,p}} \, \mathrm{d}r\right]. \quad (18)$$

Consider $(\rho_n, n \geq 1)$ a one-dimensional positive mollifier, we can define $\mathrm{P} \otimes \mathrm{d}r$ a.s., $\tilde{D}_T u(s)$ by

$$\tilde{D}_T u(r) = \lim_{n\to\infty} \int_0^T \rho_n(\tau) \mathcal{K}_T^*(\nabla_r u)(\tau - r)\, \mathrm{d}\tau.$$

Hence, $\tilde{D}_T u(r)$ is measurable with respect to (ω, r) and according to (18), we have (16). Substituting 0 to s (18), we get

$$\mathrm{E}\left[\int_0^T |\tilde{D}_T u(r)|^p \, \mathrm{d}s\right] \leq c\|u\|^p_{\mathbb{D}_{p,1}(I^{1^-}_{\sigma,p})}.$$

This means that $\int_0^{\cdot} \tilde{D}_T u(s)\, \mathrm{d}s$ belongs to $I^+_{1,p}$ and that (17) holds. □

Example 1 cont'd. In this case, $\mathcal{K}_1^* = I^{H-1/2}_{1^-}$ is continuous from $I^{1^-}_{\sigma,p}$ into $I^+_{\sigma+\alpha-1/2,p}$. This latter space is embedded in a space of Hölderian functions provided that $\sigma > 1/2 - \alpha + 1/p$. ⊐

Example 2 cont'd. According to [SKM93], $\mathcal{K}_1^* = x^{1/2-H} I_{1^-}^{H-1/2} x^{H-1/2}$ and since since $2(1+H-1/2) = 2H+1 > 1$, we infer from [SKM93, Lemma 10.1] that $\mathcal{K}_1^*$ is continuous from $I_{\sigma,p}^{1^-}$ into $I_{\sigma+\alpha-1/2,p}^{+}$, for any $\sigma \geq 0$. $\sqsupset$

Theorem 4.2. *Let $\alpha < 1/2$ and $p \geq 2$. Assume that Hypothesis I and II hold. Assume furthermore that there exists $\sigma > 1/p$ and $\eta > 0$, such that $\mathcal{K}_1^*$ is continuous from $I_{\sigma,p}^{1^-}$ into $\mathrm{Hol}(\eta)$. If u belongs to $\mathbb{D}_{p,1}(I_{\sigma+\varepsilon,p}^{1^-})$, for some $\varepsilon > 0$, then u is Stratonovitch integrable on $[0,T]$ for any $T \in [0,1]$, and*

$$\int_0^T u(s) \circ \, dX_s = \delta(\mathcal{K}_T^* u) + \int_0^T \tilde{D}_T u(s) \, ds + u(T)X(T). \tag{19}$$

Proof. For the latest sum of $R_T^\pi(u - u(T))$, we have according to Theorem 4.1,

$$\begin{aligned} \mathrm{E}\left[\left|\sum_{t_i \in \pi} \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} \mathcal{K}_T^* \nabla_r (u - u(T))(s) \, \mathrm{d}s \, \mathrm{d}r - \int_0^T \tilde{D}_T u(r) \, \mathrm{d}r \right|^p\right] \\ \leq c \, \mathrm{E}\left[\sum_{t_i \in \pi} \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} |\mathcal{K}_T^* (\nabla_r (u - u(T))(s) - \tilde{D}_T u(r)|^p \, \mathrm{d}s \, \mathrm{d}r\right] \\ \leq c \, \mathrm{E}\left[\sum_{t_i \in \pi} \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} |s - r|^{p\eta} \|\nabla_r u\|_{I_{\sigma+\varepsilon,p}^{1^-}}^p \, \mathrm{d}s \, \mathrm{d}r\right] \\ \leq c \, |\pi|^{p\eta} \|u\|_{\mathbb{D}_{p,1}(I_{\sigma+\varepsilon,p}^{1^-})}^p. \end{aligned}$$

Therefore, the latest sum of $R_T^\pi(u - u(T))$ converges in $L^p(\Omega)$ (and thus in probability) to $\int_0^T \tilde{D}_T u(s) \, \mathrm{d}s$. In order to conclude, note that in virtue of the continuity of the divergence, the first term of $R_T^\pi(u - u(T))$ tends to $\delta(\mathcal{K}_T^*(u - u(T)))$, see [Nua95]. □

Lemma 4.2. *Under the assumptions of Theorem 4.2, for any $0 \leq S \leq T \leq 1$, $u\mathbf{1}_{[0,S]}$ is Stratonovitch integrable on $[0,T]$ and we have*

$$\int_0^T (u(r) - u(S))\mathbf{1}_{[0,S]}(r) \circ \, dX_r = \int_0^S u(r) \circ \, dX_r, \tag{20}$$

for any $0 \leq S \leq T \leq 1$.

Proof. According to Eqn. (12) and to Lemma 4.1, we have

$$R_T^\pi(p_S(u - u(S))) = R_S^\pi(u - u(S)) + u(S)X^\pi(S).$$

According to Theorem 4.2, the right-hand-side sum converges so that $u\mathbf{1}_{[0,S]}$ is Stratonovitch integrable on $[0,T]$ and Eqn. (20) follows by remarking that $p_S(u - u(S))(T) = 0$. □

Remark 4.4. For the hypothesis " $\mathcal{K}_1^*$ is continuous from $I^{1^-}_{\sigma,p}$ into $\mathrm{Hol}(\eta)$" to hold, in view of the examples cited above, this requires that σ to be greater than $1/2-\alpha+1/p+\eta$.

For $\alpha>1/2$, the map $\mathcal{K}$ is still a regularizing operator so that the hypothesis are much weaker. Following the very same lines, we can prove:

Theorem 4.3. *Let $\alpha>1/2$. Assume that Hypothesis I holds. Assume furthermore that $\mathcal{K}_1^*$ is continuous from $\mathcal{L}^p$ into $I^-_{\alpha-1/2,p}$ for some $p>(\alpha-1/2)^{-1}$. If u belongs to $\mathbb{D}_{p,1}(\mathcal{L}^p)$, then, for any $T\in[0,1]$, there exists a measurable and integrable process, denoted by $\tilde{D}_T u$ such that, for almost any r,*

$$E\left[|\nabla_r\mathcal{K}_T^*u(s)-\tilde{D}_Tu(r)|^p\right]^{1/p}\le c\,|s-r|^{\alpha-1/2-1/p}\|\nabla_r u\|_{L^p(\Omega\times[0,1])}.$$

Moreover,

$$E\left[\|\int_0^\cdot \tilde{D}u(r)\,dr\|^p_{\mathrm{Hol}(1-1/p)}\right]\le c\|u\|^p_{\mathbb{D}_{p,1}(\mathcal{L}^p)}.$$

Theorem 4.4. *Assume that Hypothesis I holds for $\alpha>1/2$. Assume furthermore that $\mathcal{K}_1^*$ is continuous from $\mathcal{L}^p$ into $I^-_{\alpha-1/2,p}$ for some $p>(\alpha-1/2)^{-1}$. If u belongs to $\mathbb{D}_{p,1}(\mathcal{L}^p)$, then for any $T\in[0,1]$, u is Stratonovitch integrable on $[0,T]$ and*

$$\int_0^T u_s\circ\,dX_s=\delta(\mathcal{K}_T^*u)+\int_0^T\tilde{D}_Tu(s)\,ds.$$

Remark 4.5. The difference in this case is that $\mathcal{L}^p([0,1])$ is stable by the maps p_T so that we immediatly have:

$$\int_0^T u(s)\circ\,\mathrm{d}X_s=\int_0^1 u(s)\mathbf{1}_{[0,T]}(s)\circ\,\mathrm{d}X_s,$$

in both theorems 4.2 and 4.4.

Coming back to $\mathrm{SS}_\pi(u)$, we have:

$$\mathrm{SS}_\pi(u)=\delta\Bigg(\sum_{t_i\in\pi}\frac{1}{\delta_i}\int_{t_i}^{t_{i+1}}u_s\,\mathrm{d}s\,\big(K(t_{i+1},.)-K(t_i,.)\big)\Bigg)\\ +\sum_{t_i\in\pi}\frac{1}{\delta_i}\int_{t_i}^{t_{i+1}}\Big(K(\nabla.u_s)(t_{i+1})-K(\nabla.u_s)(t_i)\Big)\,\mathrm{d}s$$

The trace-like term is similar to those we had to treat in the previous theorems. The difference is that its limit is formally $\int_0^1(\mathcal{K}\nabla)_s u(s)\,\mathrm{d}s$ instead of $\int_0^1\nabla(\mathcal{K}_1^*u)(s)\,\mathrm{d}s$ in Theorems 4.1 and 4.3. We thus need some regularity of the map $s\mapsto\nabla_s u(r)$ which is something less easy to verify than properties on the map $s\mapsto\nabla_r u(s)$. This restriction reduces the interest of this approach.

Theorem 4.5. *Assume that Hypothesis I holds for $\alpha > 1/2$. Assume furthermore that $\mathcal{K}$ is continuous from $\mathcal{L}^p([0,1])$ into $I^-_{\alpha-1/2,p}$ for some $p > (\alpha - 1/2)^{-1}$. If u belongs to $\mathbb{D}_{p,1}(\mathcal{L}^p([0,1]))$, then there exists a measurable and integrable process, denoted by $\hat{D}u$ such that, for almost any r,*

$$E\left[|(\mathcal{K}\nabla)_s u(r) - \hat{D}u(r)|\right] \leq c\,|s-r|^\eta \|D_. u(r)\|_{\mathbb{D}_{p,1}(\mathcal{L}^p([0,1]))}. \tag{21}$$

Moreover,

$$E\left[\|\int_0^. \hat{D}u(r)\; dr\|^p_{\mathrm{Hol}(1-1/p)}\right] \leq c\|u\|^p_{\mathbb{D}_{p,1}(\mathcal{L}^p([0,1]))}. \tag{22}$$

Furthermore, $\mathcal{K}_T^* u$ belongs to $\operatorname{Dom}\delta$ and the family $SS_\pi(u)$ converges in $L^2(\Omega)$ to $\delta(\mathcal{K}_T^* u) + \int_0^T \hat{D}u(s)\; ds$.

Remark 4.6. For u belonging to $\mathbb{D}_{p,1}(\mathcal{L}^p([0,1]))$ and cylindric, it is easy to see that

$$\int_0^1 \hat{D}u(r)\; \mathrm{d}r = \int_0^1 \hat{D}_1 u(r)\; \mathrm{d}r. \tag{23}$$

According to (22) and (4.3), this remains true for any $u \in \mathbb{D}_{p,1}(\mathcal{L}^p([0,1]))$.

Remark 4.7. For $\alpha < 1/2$, one could also state a similar theorem but it would be practically of little use since it is rather hard to determine whether

$$\mathrm{E}\left[\int_0^1 \|\nabla_. u(s)\|^2_{\mathcal{S}_{1+\eta-\alpha,2}}\; \mathrm{d}s\right] \text{ is finite.}$$

5 Regularity

There are two kinds of regularity results which may be interesting : continuity with respect to the time variable and continuity with respect to the kernel. Actually, when one thinks to the generalized fBm (see Example 3), the complete identification of the model requires the perfect knowledge of the function H. Since that seems out of reach, one can naturally ask how much an error on H will modify the stochastic integral of a given integrand. The trace-like term can be controlled via theorems 4.3 and 4.1. We are now interested in the divergence part. We denote by $\|\mathcal{K}_1^*\|_{\alpha,p}$, the norm of $\mathcal{K}_1^*$ as a map from $I'_{\alpha-1/2,p}$ into $\mathcal{L}^p$.

Theorem 5.1. *Let $\alpha \in (0,1/2)$ and $1 < p < (1/2-\alpha)^{-1}$, assume that assumptions I and II hold. Assume furthermore that there exists $\varepsilon \in (0, 1/p - (1/2-\alpha))$ such that u belongs to $\mathbb{D}_{p,1}(I_{1/2-\alpha+\varepsilon,p})$. Then, the process $\{\delta(\mathcal{K}_t^* u), t \in [0,1]\}$ admits a modification with $\tilde{\varepsilon}$-Hölder continuous paths for any $\tilde{\varepsilon} < \varepsilon$, and we have the maximal inequality :*

$$\|\delta(\mathcal{K}_.^* u)\|_{L^p(\Omega; I^+_{1/p^*+\tilde{\varepsilon},p^*})} \leq c\,\|\mathcal{K}^*\|_{\alpha,p}\|u\|_{\mathbb{D}_{p,1}(I_{1/2-\alpha+\varepsilon,p})}.$$

Proof. Since $1/2-\alpha+\varepsilon$ is strictly less than $1/p$, we know that for any $T\in[0,1]$, $p_T u$ belongs to $I_{1/p-\alpha+\varepsilon,p}$, see Proposition 2.1. In view of Lemma 4.1, we have $\delta(\mathcal{K}_t^* u)=\delta(\mathcal{K}_1^*(u\mathbf{1}_{[0,t]}))$. Therefore, for $g\in\mathcal{C}^\infty$ and ψ a cylindric real-valued functional,

$$\begin{aligned}\mathrm{E}\left[\int_0^1 \delta\mathcal{K}_1^*(u\mathbf{1}_{[0,t]})g(t)\,\mathrm{d}t\,\psi\right]&=\mathrm{E}\left[\iint_{[0,1]^2}\mathcal{K}_1^*(u\mathbf{1}_{[0,t]})(r)g(t)\nabla_r\psi\,\mathrm{d}t\,\mathrm{d}r\right]\\&=\mathrm{E}\left[\int_0^1\mathcal{K}_1^*(uI^1_{1^-}g)(r)\nabla_r\psi\,\mathrm{d}r\right]=\mathrm{E}\left[\delta(\mathcal{K}_1^*(u.I^1_{1^-}g)\psi\right].\end{aligned}$$

Thus,

$$\int_0^1\delta(\mathcal{K}_t^* u)g(t)\,\mathrm{d}t=\delta(\mathcal{K}_1^*(u.I^1_{1^-}g))\ \text{P-a.s..}\tag{24}$$

Since $p<(1/2-\alpha)^{-1}$, $1/2-\alpha<1/p$, we can then apply Corollary 2.1 with $t=1/2-\alpha$, $r=1/p-\tilde{\varepsilon}$ and $s=1/2-\alpha+\varepsilon$. Since g is deterministic, we have

$$\|\delta(\mathcal{K}_1^*(u.I^1_{1^-}g))\|_{L^p(\Omega)}\le c\|\mathcal{K}^*\|_{\alpha,p}\|u\|_{\mathbb{D}_{p,1}(I_{1/2-\alpha+\varepsilon,p})}\|I^1_{1^-}g\|_{I_{1/p-\tilde{\varepsilon},p}}.\tag{25}$$

We then obtain that for $\psi\in L^{p^*}(\Omega)$, for $g\in(I^-_{1/p-1-\tilde{\varepsilon},p})'$,

$$\begin{aligned}&|\mathrm{E}\left[\int_0^1\delta\mathcal{K}_1^*(u\mathbf{1}_{[0,t]})g(t)\,\mathrm{d}t\,\psi\right]|\\&\qquad\le c\|\mathcal{K}^*\|_{\alpha,p}\|\psi\|_{L^{p^*}(\Omega)}\|g\|_{(I^-_{1/p-1-\tilde{\varepsilon},p})'}\|u\|_{\mathbb{D}_{2,1}(\mathcal{S}_{1/2-\alpha,p})}.\end{aligned}\tag{26}$$

It follows that $\{\delta(\mathcal{K}_t^* u),\,t\in[0,1]\}$ belongs to $(L^{p^*}(\Omega;I^-_{-1+1/p-\tilde{\varepsilon},p}))'$, which is isomorphic to $L^p(\Omega;I^+_{1-1/p+\tilde{\varepsilon},p^*})$, and that

$$\|\delta(\mathcal{K}_.^* u)\|_{L^p(\Omega;I^+_{1/p^*+\tilde{\varepsilon},p^*})}\le c\,\|\mathcal{K}^*\|_{\alpha,p}\|u\|_{\mathbb{D}_{p,1}(I_{1/2-\alpha+\varepsilon,p})}.$$

This induces that there exists a modification of $\{\delta(\mathcal{K}_t^* u),\,t\in[0,1]\}$ with $\tilde{\varepsilon}$-Hölder continuous sample-paths. □

Remark 5.1. Note that 1 belongs to $I_{1/2-\varepsilon,2}$ for any $\varepsilon>0$, thus we retrieve that $X_t=\delta(\mathcal{K}_1^* p_t u)$ has a version with $(\alpha-\varepsilon)$-Hölder continuous sample-paths.

If $\varepsilon>1/p-1/2+\alpha$, we cannot apply Lemma 2.1 any more, since $s=1/2-\alpha+\varepsilon$ would be greater than $1/p$. This is more than a technical problem: in this situation, i.e., $u\in I_{\varepsilon+1/2-\alpha,p}$, u is continuous and $p_T u$ does not necesssary belongs to $I_{\varepsilon+1/2-\alpha,p}$, so that the whole principle of the above proof fails. However, as Lemma 2.3 shows, if we consider $p_T(u-u(T))$ instead of $P_T u$, this function belongs to $I_{\varepsilon+1/2-\alpha,p}$, for a smaller ε. Thus, we have:

Theorem 5.2. *Let* $\alpha \in (0, 1/2)$ *and* $p > 1$, *assume that assumptions I and II hold. Assume furthermore that there exists* $\varepsilon \in ((1/p - 1/2 + \alpha)^+, 1)$ *such that u belongs to* $\mathbb{D}_{p,1}(I^-_{\varepsilon+1/2-\alpha,p})$. *Then, for any* $\tilde{\varepsilon} < \varepsilon$, *the process* $\{\delta(\mathcal{K}_t^*(u - u(t))), t \in [0,1]\}$ *admits a modification with* $\tilde{\varepsilon}$*-Hölder continuous paths and we have the maximal inequality :*

$$\|\delta(\mathcal{K}_.^*(u - u(.)))\|_{L^p(\Omega; I^+_{1/p^*+\tilde{\varepsilon},p^*})} \le c\, \|\mathcal{K}^*\|_{\alpha,p} \|u\|_{\mathbb{D}_{p,1}(I^-_{\varepsilon+1/2-\alpha,p})}. \tag{27}$$

Proof. Note that we are allowed to consider $u - u(t)$ since $1/p - 1/2 + \alpha < \varepsilon$ implies that $\varepsilon + 1/2 - \alpha > 1/p$ and thus that $I^-_{\varepsilon+1/2-\alpha,p}$ is embedded in $\mathrm{Hol}(\varepsilon + 1/2 - \alpha - 1/p)$. The very same techniques as above show that

$$\int_0^1 \delta(\mathcal{K}_t^*(u - u(t)))g(t)\ \mathrm{d}t = \delta(\mathcal{K}_1^*(uI^1_{1^-}g - I^1_{1^-}(ug))),\ \mathrm{P\ a.s.}.$$

A classical integration by parts and then a fractional integration by parts (see (2)) give that

$$\int_0^1 \delta(\mathcal{K}_t^*(u - u(t)))g(t)\ \mathrm{d}t = -\delta(\mathcal{K}_1^*(I^1_{1^-}(I^{-\zeta}_{0^+}uI^{\zeta}_{1^-}g))),\ \mathrm{P\ a.s.}.$$

Now, we clearly have

$$\|I^1_{1^-}(I^{-\zeta}_{0^+}uI^{\zeta}_{1^-}g)\|_{I_{1/2-\alpha,p}} = \|I^{-\zeta}_{0^+}uI^{\zeta}_{1^-}g\|_{I_{-1/2-\alpha,p}}.$$

Applying Corollary 2.1 with $\zeta = 1/2 - \alpha + \varepsilon - 1/p + \varepsilon'$, $t = -(1/2 + \alpha)$, $s + \zeta = 1/2 - \alpha + \varepsilon$ and $r + s = t + 1/p + \varepsilon'$ for some $\varepsilon' > 0$ sufficiently small, we get

$$\begin{aligned}
\|\mathcal{K}_1^*(I^1_{1^-}(I^{-\zeta}_{0^+}uI^{\zeta}_{1^-}g))\|_{\mathcal{L}^p} &\le c\, \|I^{-\zeta}_{0^+}u\|_{I^-_{s,p}} \|I^{\zeta}_{1^-}g\|_{I^-_{r,p}} \\
&= c\, \|u\|_{I^-_{s+\zeta,p}} \|g\|_{I^-_{r-\zeta,p}} \\
&= c\, \|u\|_{I^-_{1/2-\alpha+\varepsilon,p}} \|g\|_{I^-_{-1+1/p-\varepsilon+\varepsilon',p}}.
\end{aligned}$$

It follows as in the previous proof that $\{\delta(\mathcal{K}_t^*(u - u(t))), t \ge 0\}$ belongs to $L^p(\Omega;\ I^+_{1/p^*+\tilde{\varepsilon},p^*})$ (with $\tilde{\varepsilon} = \varepsilon - \varepsilon'$) and that the maximal inequality (27) holds. □

Theorem 5.3. *For any* $\alpha \in [1/2, 1)$, *assume that assumption I holds. Let u belong to* $\mathbb{D}_{p,1}(\mathcal{L}^p)$ *with* $\alpha p > 1$. *The process* $\{\delta(\mathcal{K}_t^* u), t \in [0,1]\}$ *admits a modification with* $(\alpha - 1/p)$*-Hölder continuous paths and we have the maximal inequality :*

$$\|\delta(\mathcal{K}_.^* u)\|_{L^p(\Omega; \mathrm{Hol}(\alpha - 1/p))} \le c \|\mathcal{K}_1^*\|_{\alpha,2} \|u\|_{\mathbb{D}_{p,1}}.$$

Proof. We begin as in Theorem 5.1 until Eqn. (24). Since $\alpha > 1/2$, it is clear that $\mathcal{K}$ is continuous from $\mathcal{L}^2([0,1])$ into $I_{\alpha-1/2,2}$ thus that $\mathcal{K}^*$ is continuous from $I'_{\alpha-1/2,2}$ in $\mathcal{L}^2([0,1])$. Since $I_{\alpha-1/2,2}$ is continuously embedded in $\mathcal{L}^{(1-\alpha)^{-1}}$, it follows that $\mathcal{L}^{1/\alpha} = (\mathcal{L}^{1/(1-\alpha)})'$ is continuously embedded in $I_{1/2-\alpha,2}$. Since u belongs to $\mathbb{D}_{p,1}(\mathcal{L}^p)$, the generalized Hölder inequality implies that

$$\|uI^1_{1^-}g\|_{\mathcal{L}^{1/\alpha}} \le \|u\|_{\mathcal{L}^p} \|I^1_{1^-}g\|_{\mathcal{L}^{(\alpha-1/p)^{-1}}}.$$

It follows that $\{\delta(\mathcal{K}_t^*u), t \in [0,1]\}$ belongs to $L^p(\Omega; I^+_{1,(1-\alpha+1/p)^{-1}})$ with

$$\|\delta(\mathcal{K}_.^*u)\|_{L^p(\Omega; I^+_{1,(1-\alpha+1/p)^{-1}})} \le c\|\mathcal{K}_1^*\|_{\alpha,2}\|u\|_{\mathbb{D}_{p,1}}.$$

The proof is completed remarking that $1-1/(1-\alpha+1/p)^{-1} = \alpha - 1/p$ so that $I^+_{1,(1-\alpha+1/p)^{-1}}$ is embedded in $\mathrm{Hol}(\alpha - 1/p)$. □

Remark 5.2. These results extend similar results in [AMN01] in the sense that the assumptions on the kernel and on the integrand are here much weaker for the same conclusion.

6 Itô Formula

We are now interested in non-linear transformations of Itô-like processes:

$$Z(t) = z + \int_0^t u(s) \circ \, \mathrm{d}X_s, \tag{28}$$

for a sufficiently regular u. The Itô formula for fBm-like processes has already a long history. There are two technical barriers: it is relatively easy to prove Itô formula for $\alpha > 1/2$, since we then have a process more regular than the ordinary Brownian motion and all the limiting procedures are straightforward (cf. [DH96, DÜ95, DÜ99]) . Harder is the situation where α belongs to $(0, 1/2]$, Alòs et al. [ALN01] obtained a formula for the fBm of Hurst index greater than $1/4$. By a very different procedure, Gradinaru et al. [GRV] were able to include $1/4$ in the domain of validity of the formula. In another different approach, Feyel et al. [FdlP01] also gave a formula for any Hurst index via analytic continuation of the formula obtained for $\alpha \ge 1/2$. Carmona et al. [CCM02] obtained an Itô formula for $\alpha > 1/6$, for a class of processes similar to our so-called Volterra processes.

The following results owes much to the paper [CCM02] which shows that it was possible to go beyond the barrier $1/4$, to the paper [AMN01] which gives the simplest expression of the Itô formula and to the work [GRV] which emphasizes the importance of symmetrization. Actually, the key remark is that there exists integrands u for which

$$\begin{aligned} R_h(u) &:= h^{-1}\int_0^1 \Big(\mathcal{K}_1^* p_{t+h}u(s) - \mathcal{K}_1^* p_t u(s)\Big)\Big(\mathcal{K}_1^* p_{t+h}u(s) + \mathcal{K}_1^* p_t u(s)\Big)\,\mathrm{d}s \\ &= h^{-1}\int_0^1 \mathcal{K}_1^* p_{t,t+h}u(s)\,\mathcal{K}_1^*(p_t + p_{t+h})u(s)\,\mathrm{d}s \\ &= h^{-1}\int_0^1 (\mathcal{K}_1^* p_{t+h}u(s)^2 - \mathcal{K}_1^* p_t u(s)^2)\,\mathrm{d}s, \end{aligned} \tag{29}$$

has a finite limit. If $u \equiv 1$, since $\mathbf{1}_{[0,t)} = I_1^*(\varepsilon_t)$, it follows from the definition of $\mathcal{K}$ that $\mathcal{K}_1^* p_t \mathbf{1} = K(t,.)$ and thus $R_h(\mathbf{1}) = h^{-1}(R(t+h,t+h) - R(t,t))$, where R is the covariance kernel of X. For instance, if X is the fBm with stationary increments, this expression is proportional to $h^{-1}((t+h)^{2\alpha} - t^{2\alpha})$. The different barriers can be explained from the behavior of this last term, whose limit is clearly $t^{2\alpha-1}$. When $\alpha > 1/2$, this is a bounded function of t so easily controllable in the limiting procedures. For $\alpha \in (1/4, 1/2)$, it is no longer bounded but still in $\mathcal{L}^2([0,1])$. When, $\alpha < 1/4$, we only have an $\mathcal{L}^p$ integrable function for $1 - p^{-1} < 2\alpha$.

Hypothesis III. Let $\mathcal{R}$ the set of processes such that $R_h(u)$, as defined in (29), has a finite limit in $L^1(\Omega)$. We assume that $\mathcal{K}_1^*$ is such that $\mathcal{R}$ is non-empty.

Lemma 6.1. *Let $\alpha \in (0,1)$, be given and assume that hypothesis I, II andIII hold. Let u be a cylindric process, belonging to $\mathcal{R}$. Let*

$$n_\alpha := \inf\{n:\ 2n\alpha > 1\}.$$

For any $f \in C_b^{n_\alpha}$, i.e., n_α-times differentiable with bounded derivatives, we have

$$\begin{aligned} \frac{d}{dt}E\Big[f(Z_t)\psi\Big] &= E\Big[f'(Z_t)(\mathcal{K}\nabla)_t(u(t)\psi)\Big] \\ &\quad + \frac{1}{2}E\Big[f''(Z_t)\psi\frac{d}{dt}\int_0^1 \mathcal{K}_1^*(p_t u)(s)^2\,ds\Big] \\ &\quad + E\Big[u(t)f''(Z_t)\psi(\mathcal{K}\nabla)_t\Big(\int_0^t (\mathcal{K}\nabla)_r u(r)\,dr\Big)\Big] \\ &\quad + E\Big[u(t)f''(Z_t)\delta\big((\mathcal{K}\nabla)_t(\mathcal{K}_1^* p_t u)\big)\psi\Big]. \end{aligned} \tag{30}$$

Proof. Introduce the function g as

$$g(x) = f(\frac{a+b}{2} + x) - f(\frac{a+b}{2} - x).$$

This function is even, satisfies

$$g^{(2j+1)}(0) = 2f^{(2j+1)}((a+b)/2) \text{ and } g(\frac{b-a}{2}) = f(b) - f(a).$$

Applying the Taylor formula to g between the points 0 and $(b-a)/2$, we get

$$f(b) - f(a) = \sum_{j=0}^{n-1} \frac{2^{-2j}}{(2j+1)!}(b-a)^{2j+1} f^{(2j+1)}(\frac{a+b}{2}) \\ + \frac{(b-a)^{2n}}{(2n)!} \int_0^1 \lambda^{2n-1} g^{(2n)}(\lambda a + (1-\lambda)b) \, \mathrm{d}\lambda.$$

We thus have

$$\mathrm{E}\left[(f(Z_{t+h}) - f(Z_t))\psi\right] = \sum_{j=0}^{n_\alpha - 1} \frac{2^{-2j}}{(2j+1)!} \mathrm{E}\left[(b-a)^{2j+1} f^{(2j+1)}(\frac{a+b}{2}) \, \psi\right] \\ + \frac{1}{2n_\alpha!} \mathrm{E}\left[(Z_{t+h} - Z_t)^{(2n_\alpha)} \int_0^1 r^{2n_\alpha - 1} g^{(2n_\alpha)}(rZ_t + (1-r)Z_{t+h}) \, \mathrm{d}r \, \psi\right]. \quad (31)$$

We need to prove that, when divided by h, the latter quantity has a limit when h goes to 0. It turns out that the sole contributing term is the first one. We first show that n_α is chosen sufficiently large so that the last term vanish. Since Z belongs $L^2(\Omega; \mathrm{Hol}(\alpha - \varepsilon))$ for any $\varepsilon > 0$, and since $g^{(2n_\alpha)}$ is bounded, the last term is bounded by a constant times $h^{2n_\alpha(\alpha-\varepsilon)}$. Hence, this last term divided by h vanishes when h goes to 0. We next deal with the first order term. Since u is cylindric,

$$Z_t = \delta(\mathcal{K}_1^* p_t u) + \int_0^t \mathcal{K}_1^*(\nabla_s p_t u)(s) \, \mathrm{d}s. \quad (32)$$

Substitute Eqn. (32) into the first order term and use integration by parts formula, this yields to:

$$
\begin{aligned}
&\mathrm{E}\left[(Z_{t+h}-Z_t)\,f'(\frac{Z_t+Z_{t+h}}{2})\psi\right]\\
&=\mathrm{E}\left[\int \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s(\,f'(\frac{Z_t+Z_{t+h}}{2})\psi)\ \mathrm{d}s\right]\\
&\qquad\qquad\qquad\qquad+\mathrm{E}\left[f'(\frac{Z_t+Z_{t+h}}{2})\psi\int \mathcal{K}_1^*(p_{t,t+h}\nabla_s u)(s)\ \mathrm{d}s\ \right]\\
&=\mathrm{E}\left[f'(\frac{Z_t+Z_{t+h}}{2})\int \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s\psi\ \mathrm{d}s\ \right]\\
&\qquad\qquad+\mathrm{E}\left[f''(\frac{Z_t+Z_{t+h}}{2})\psi\int \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s(\frac{Z_t+Z_{t+h}}{2})\ \mathrm{d}s\ \right]\\
&\qquad\qquad+\mathrm{E}\left[f'(\frac{Z_t+Z_{t+h}}{2})\psi\int_0^1 \mathcal{K}_1^*(p_{t,t+h}\nabla_s u)(s)\ \mathrm{d}s\ \right]=\sum_{i=1}^3 A_i.
\end{aligned}
$$

We can write A_1 as

$$
A_1=\mathrm{E}\left[\int_t^{t+h}u(s)(\mathcal{K}\nabla)_s\psi\ \mathrm{d}s\ f'(\frac{Z_t+Z_{t+h}}{2})\psi\right],
$$

by dominated convergence, it is then easily shown that

$$
\lim_{h\to 0}h^{-1}A_1=\mathrm{E}\left[u(t)\,f'(Z_t)(\mathcal{K}\nabla)_t\psi\right]. \tag{33}
$$

By direct calculations, since u is cylindric, we have

$$
\int_0^1 \mathcal{K}_1^*(p_{t,t+h}\nabla_s u)(s)\ \mathrm{d}s=\int_t^{t+h}(\mathcal{K}\nabla)_s u(s)\ \mathrm{d}s,\ \text{thus,}
$$

$$
\lim_{h\to 0}h^{-1}A_3=\mathrm{E}\left[\,f'(Z_t)\psi(\mathcal{K}\nabla)_t u(t)\right]. \tag{34}
$$

Expanding $\nabla_s(Z_t+Z_{t+h})$, we obtain

$$
\begin{aligned}
2A_2&=\mathrm{E}\left[f''(\frac{Z_t+Z_{t+h}}{2})\psi\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\,\mathcal{K}_1^*(p_t u+p_{t+h}u)(s)\ \mathrm{d}s\ \right]\\
&+\mathrm{E}\left[f''(\frac{Z_t+Z_{t+h}}{2})\psi\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\delta\Big(\mathcal{K}_1^*(p_t+p_{t+h})\nabla_s u)\Big)\ \mathrm{d}s\ \right]\\
&+\mathrm{E}\left[\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s\Big(\int_0^1(p_t+p_{t+h})(\mathcal{K}\nabla)_r u(r)\ \mathrm{d}r\ \mathrm{d}r\Big)\ \mathrm{d}s\right.\\
&\qquad\qquad\qquad\qquad\left.\times\ f''(\frac{Z_t+Z_{t+h}}{2})\psi\right]=\sum_{i=1}^3 B_i.
\end{aligned}
$$

According to Hypothesis III,

$$\lim_{h\to 0} h^{-1}B_1 = \mathrm{E}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mathcal{K}_1^*(p_t u)(s)^2 \,\mathrm{d}s\, f''(Z_t)\psi\right]. \tag{35}$$

It is rather clear that

$$\lim_{h\to 0} h^{-1}B_3 = 2\mathrm{E}\left[u(t)(\mathcal{K}\nabla)_t(\int_0^t (\mathcal{K}\nabla)_r u(r)\,\mathrm{d}r) f''(Z_t)\psi\right]. \tag{36}$$

To deal with B_2, we need to apply once more the integration by parts formula. This gives,

$$\begin{aligned}
B_2 = \mathrm{E}\Bigg[&\int_0^1\int_0^1 \mathcal{K}_1^*(p_{t,t+h}\nabla_r u)(s)\nabla_s \mathcal{K}_1^*(p_t + p_{t+h})u(r)\,\mathrm{d}s\,\mathrm{d}r \\
&\qquad\qquad\qquad\qquad \times f''(\frac{Z_t+Z_{t+h}}{2})\psi\Bigg] \\
+\mathrm{E}\Bigg[&\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\int_0^1 \mathcal{K}_1^*((p_t+p_{t+h})\nabla_s u)(r) \\
&\qquad\qquad\qquad \times\nabla_r(f''(\frac{Z_t+Z_{t+h}}{2})\psi)\,\mathrm{d}r\,\mathrm{d}s\Bigg].
\end{aligned}$$

It follows from this expression that

$$\begin{aligned}
\lim_{h\to 0} h^{-1}B_2 &= 2\,\mathrm{E}\left[\int_0^1 (\mathcal{K}\nabla)_t(\mathcal{K}_1^* p_t u)(r)\nabla_r(u(t)f''(Z_t)\psi)\,\mathrm{d}r\right] \\
&= 2\,\mathrm{E}\left[u(t)f''(Z_t)\psi\,\delta\big((\mathcal{K}\nabla)_t\mathcal{K}_1^* p_t u\big)\right].
\end{aligned} \tag{37}$$

The remaining terms are of the form

$$\begin{aligned}
&\mathrm{E}\left[(Z_{t+h}-Z_t)^{2j+1} f^{(2j+1)}(\frac{Z_t+Z_{t+h}}{2})\psi\right] \\
&\quad = \mathrm{E}\left[\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s\Big((Z_{t+h}-Z_t)^{2j} f^{(2j+1)}(\frac{Z_t+Z_{t+h}}{2})\psi\Big)\,\mathrm{d}s\right] \\
&\;+\mathrm{E}\left[(Z_{t+h}-Z_t)^{2j} f^{(2j+1)}(\frac{Z_t+Z_{t+h}}{2})\psi\int_0^1 \mathcal{K}_1^*(p_{t,t+h}\nabla_s u)(s)\,\mathrm{d}s\right] = C_1 + C_2.
\end{aligned}$$

By dominated convergence, it is clear that $h^{-1}C_2$ vanishes as h goes to 0. As to C_1, it can be splitted into three parts

$$
\begin{aligned}
C_1 = 2j\,\mathrm{E}\Bigg[&(Z_{t+h}-Z_t)^{2j-1} f^{(2j+1)}(\frac{Z_t+Z_{t+h}}{2})\psi \\
&\times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s(Z_{t+h}-Z_t)\,\mathrm{d}s\Bigg] \\
+\mathrm{E}\Bigg[&(Z_{t+h}-Z_t)^{2j} f^{(2j+2)}(\frac{Z_t+Z_{t+h}}{2})\psi \\
&\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s(\frac{Z_t+Z_{t+h}}{2})\,\mathrm{d}s\Bigg] \\
+\mathrm{E}\Bigg[&(Z_{t+h}-Z_t)^{2j} f^{(2j+1)}(\frac{Z_t+Z_{t+h}}{2}) \\
&\times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s\psi\,\mathrm{d}s\Bigg] = \sum_{i=1}^3 D_i.
\end{aligned}
$$

By dominated convergence, $h^{-1}D_3$ vanishes as h goes to 0. Expanding the Gross-Sobolev derivative D_2, we get

$$
\begin{aligned}
2D_2 = \mathrm{E}\Bigg[&f^{(2j+2)}(\frac{Z_t+Z_{t+h}}{2})(Z_{t+h}-Z_t)^{2j}\psi \\
&\times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\mathcal{K}_1^*(p_t u + p_{t+h}u)(s)\,\mathrm{d}s\Bigg] \\
+\mathrm{E}\Bigg[&f^{(2j+2)}(\frac{Z_t+Z_{t+h}}{2})(Z_{t+h}-Z_t)^{2j}\psi \\
&\times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\delta\Big(\mathcal{K}_1^*(p_t\nabla_s u + p_{t+h}\nabla_s u)\Big)\,\mathrm{d}s\Bigg] \\
+\mathrm{E}\Bigg[&f^{(2j+2)}(\frac{Z_t+Z_{t+h}}{2})(Z_{t+h}-Z_t)^{2j}\psi \\
&\times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s\Big(\int_0^1 (p_t+p_{t+h})(\mathcal{K}\nabla)_r u(r)\,\mathrm{d}r\Big)\,\mathrm{d}s\Bigg].
\end{aligned}
$$

Following the reasoning applied to A_2, we see that all the terms in the integrals are converging a.s. (when divided by h) to a finite limit, since there still is a factor $(Z_{t+h}-Z_t)^{2j}$, with $j>0$, the product converges to 0. By dominated convergence, the convergence can be seen to hold in $L^1(\Omega)$, thus $h^{-1}D_2$ goes to 0 as h goes to 0. The really difficult term is D_1. For the sake of clarity, we only treat the case $j=1$.

For $j=1$,

$$
\begin{aligned}
D_1 &= \mathrm{E}\Bigg[(Z_{t+h}-Z_t)\, f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \\
&\qquad\qquad \times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\nabla_s(Z_{t+h}-Z_t)\ \mathrm{d}s\Bigg] \\
&= 2\ \mathrm{E}\Bigg[(Z_{t+h}-Z_t)\, f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)^2\ \mathrm{d}s\Bigg] \\
&+ 2\ \mathrm{E}\Bigg[(Z_{t+h}-Z_t)\, f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \\
&\qquad\qquad \times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\delta(\mathcal{K}_1^*((p_t+p_{t+h})\nabla_s u))\ \mathrm{d}s\Bigg] \\
&+ 2\ \mathrm{E}\Bigg[(Z_{t+h}-Z_t)\, f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \\
&\qquad\qquad \times \int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s) \int_0^1 \mathcal{K}_1^*(\nabla^{(2)}_{r,s}(p_t+p_{t+h})u)(r)\ \mathrm{d}r\ \mathrm{d}s\Bigg].
\end{aligned}
$$

Dominated convergence implies that the last term, divided by h, vanishes as h goes to 0. For the two other summands, the idea is always the same, each time there is a divergence term, we apply integration by parts formula. Then, each new term is treated by the previous methods. For instance, the most difficult term to handle is one of the term which comes from derivative of the divergence in the first summand:

$$
\begin{aligned}
&\mathrm{E}\Bigg[f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \int_0^1 \nabla_r(\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)^2\ \mathrm{d}s)\mathcal{K}_1^*(p_{t,t+h}u)(r)\ \mathrm{d}r\Bigg] \\
&= \mathrm{E}\Bigg[f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \\
&\qquad \times \int_0^1\int_0^1 \mathcal{K}_1^*(p_{t,t+h}u)(s)\mathcal{K}_1^*(p_{t,t+h}\nabla_r u)(s)\mathcal{K}_1^*(p_{t,t+h}u)(r)\ \mathrm{d}r\ \mathrm{d}s\Bigg] \\
&= \mathrm{E}\Bigg[f^{(3)}(\frac{Z_t+Z_{t+h}}{2})\Psi \\
&\qquad \times \int_t^{t+h} u(s)\mathcal{K}\Big(\int_0^1 \mathcal{K}_1^*(p_{t,t+h}\nabla_r u)(.)\mathcal{K}_1^*(p_{t,t+h}u)(r)\ \mathrm{d}r\Big)(s)\ \mathrm{d}s\Bigg].
\end{aligned}
$$

Once again, in this form, it is clear that this term, divided by h, converges to 0. All the remaining term are treated likewise and do not contribute. Thus from Eqn. (30) follows from (33), (34), (35), (36) and (37). $\square$

Since u is cylindric, all the terms of (30) are integrable with respect to t, we thus have

Corollary 6.1. *Under the assuptions of the previous lemma, we have,*

$$\begin{aligned} E\left[f(Z_t)\psi\right] &= E\left[f(x)\psi\right] + E\left[\int_0^t f'(Z_s)(\mathcal{K}\nabla)_s(u(s)\psi)\ ds\right] \\ &+ \frac{1}{2}E\left[\psi\int_0^t f''(Z_s)\frac{d}{ds}\int_0^1 \mathcal{K}_1^* p_s u(r)^2\ dr\ ds\right] \\ &+ E\left[\psi\int_0^t u(s)f''(Z_s)(\mathcal{K}\nabla)_s\Big(\int_0^s(\mathcal{K}\nabla)_r u(r)\ dr\Big)\ ds\right] \\ &+ E\left[\psi\int_0^t u(s)f''(Z_s)\delta\Big(\mathcal{K}_1^* p_s(\mathcal{K}\nabla)_s u\Big)\ ds\right], \end{aligned}$$

for any ψ *such that* $\nabla\psi$ *belongs to* Dom $\mathcal{K}$.

Since $(\mathcal{K}\nabla)$ is a derivation operator, we obtain after a few manipulations: Since $\mathcal{K}\nabla$ is a derivation operator, we have

$$\begin{aligned} \mathrm{E}\left[f(Z_t)\psi\right] &= \mathrm{E}\left[f(x)\psi\right] + \mathrm{E}\left[\int_0^t(\mathcal{K}\nabla)_s(f'(Z_s)u(s)\psi)\ \mathrm{d}s\right] \\ &+ \frac{1}{2}\mathrm{E}\left[\psi\int_0^t f''(Z_s)\frac{d}{ds}\int_0^1 \mathcal{K}_1^* p_s u(r)^2\ \mathrm{d}r\ \mathrm{d}s\right] \\ &- \mathrm{E}\left[\psi\int_0^t u(s)f''(Z_s)\mathcal{K}\mathcal{K}_1^*(p_s u)(s)\ \mathrm{d}s\right]. \end{aligned}$$

This means that for any t, we have a.e.,

$$\begin{aligned} f(Z_t) = f(x) &+ \int_0^t f'(Z_s)u(s)\circ\ \mathrm{d}X_s \\ &+ \frac{1}{2}\int_0^t f''(Z_s)\frac{d}{ds}\int_0^1 \mathcal{K}_1^* p_s u(r)^2\ \mathrm{d}r\ \mathrm{d}s \\ &- \int_0^t u(s)f''(Z_s)\mathcal{K}\mathcal{K}_1^*(p_s u)(s)\ \mathrm{d}s. \end{aligned} \tag{38}$$

Remark 6.1. It has to be noted that in [Dec02b], we announced an Itô formula for general u and any $\alpha\in(0,1)$. This is unfortunately wrong for $\alpha\in(0,1/2)$. Actually, starting from (38), the problem is now to pass to the limit. For the very first term of the righthandside of (38), we need to find a class of processes u for which $f\circ Z.u$ is Stratonovich integrable. The most restrictive part is to find conditions under which this process has a “trace” in the sense of Theorem 4.1. It is important to note that

$$\nabla_r Z_t = \mathcal{K}_1^* p_t(u-u(t))(r) + \delta(\mathcal{K}_1^* p_t \nabla_r(u-u(t))) + \nabla_r \int_0^t (\mathcal{K}\nabla)_s(u-u(t))(s)\, \mathrm{d}s \\ + X(t)\nabla_r u(t) + u(t)K(t,r)$$

and thus, we have

$$\mathcal{K}(\nabla_. Z_t)(r) = \mathcal{K}(\mathcal{K}_1^* p_t(u-u(t)))(r) + \mathcal{K}(\delta(\mathcal{K}_1^* p_t \nabla_.(u-u(t))))(r) \\ + \mathcal{K}(\nabla_. \int_0^t (\mathcal{K}\nabla)_s(u-u(t))(s)\, \mathrm{d}s)(r) + \mathcal{K}(X(t)\nabla_. u(t))(r) \\ + \mathcal{K}(u(t)K(t,.))(t).$$

It is possible to impose hypothesis on u such that the first four terms of the previous equations have a signification when $r = t$. Unfortunately, for the very last term, we have

$$\mathcal{K}(u(t)K(t,.))(t) = u(t)\frac{\partial}{\partial s}R(t,s)|_{s=t}.$$

In the case of the fBm with stationary increments, this is equal, up to a constant, to $u(t)(s^{2\alpha-1} - (t-s)^{2\alpha-1})_{s=t}$. Since this quantity is infinite for $\alpha < 1/2$, we haven't been able to go below $1/2$.

Remark 6.2. If we don't have a trace term we can state the following result.

Theorem 6.1. *Let* $\alpha \in (0,1)$, *be given and assume that hypothesis I, II andIII hold. Let u be a cylindric process, belonging to* $\mathcal{R}_.$ *Let*

$$n_\alpha := \inf\{n :\ 2n\alpha > 1\}.$$

Let

$$Z_t = \delta(\mathcal{K}_1^* p_t u).$$

For any $f \in \mathcal{C}_b^{n_\alpha}$, *i.e.,* n_α*-times differentiable with bounded derivatives, we have*

$$f(Z_t) = f(x) + \delta\Big(\mathcal{K}_t^*(u.f'\circ Z)\Big) \\ + \frac{1}{2}\int_0^t f''(Z_s)\frac{d}{ds}\int_0^1 \mathcal{K}_1^* p_s u(r)^2\, dr\, ds \\ + \int_0^t u(s) f''(Z_s)\delta\Big(\mathcal{K}_1^* p_s(\mathcal{K}\nabla)_s u\Big)\, ds,$$

for any t, *a.s..*

Proof. The proof is exactly the same as the previous one. □

If $u \equiv 1$, we get the same result as in [AMN01, CCM02, DÜ99, FdlP01] valid for any $\alpha \in (0,1)$. If $\mathcal{K} = \mathrm{Id}$, i.e., X is an ordinary Brownian motion, and u is not necessarily adapted, this formula coincides with that given in [Ü88].

7 Skorohod integral

Since the term $\int_0^T \tilde{D}_T u(s)\, \mathrm{d}s$ is a trace-like term, it is reasonable to introduce the following definitions. We now introduce a stochastic integral defined

Definition 7.1. *We denote by* $\operatorname{Dom}\delta_{\mathcal{K}^*}$, *the set of processes u belonging a.s. to* $\operatorname{Dom}\mathcal{K}^*$ *and such that* $\mathcal{K}^* u$ *belongs to* $\operatorname{Dom}\delta$. *We denote by* $\operatorname{Dom}\delta_X$, *the set of processes u in* $\operatorname{Dom}\delta_{\mathcal{K}^*}$ *such that* $\nabla\mathcal{K}^* u$ *is P-a.s. a trace class operator.*

Definition 7.2. *For* $u \in \operatorname{Dom}\delta_X$, *we define the stochastic integral of u with respect to X by*

$$\int_0^1 u_s * \, dX_s \stackrel{def}{=} \int_0^1 (\mathcal{K}^* u)(s)\delta B_s + \operatorname{trace}(\nabla(\mathcal{K}^* u))$$

To define the integral of u between time 0 and t, we use Lemma 4.1:

Definition 7.3. *For* $u \in \operatorname{Dom}\delta_X$, *we define the stochastic integral of u with respect to X between* 0 *and t by*

$$\begin{aligned}\int_0^t u_s * \, dX_s &= \int_0^1 (p_t u)(s) * \, dX_s \\ &= \int_0^t (\mathcal{K}_t^* u)(s)\delta B_s + \operatorname{trace}(p_t \nabla(\mathcal{K}_t^* u)),\end{aligned}$$

where the second equality follows by (15).

Eqn. (23) has its equivalent in this setting :

Lemma 7.1. *Assume that I andII hold. Let* $u \in \operatorname{Dom}\mathcal{K}^*$ *belong to* $\mathbb{D}_{2,1}(\mathcal{L}^2([0,1]))$ *and be such that* ∇u *belong (a.s.) to* $\operatorname{Dom}\mathcal{K}$. *Then* $\operatorname{trace}(\nabla(\mathcal{K}^* u))$ *is finite iff* $\operatorname{trace}((\mathcal{K}\nabla)u)$ *is finite and they are equal.*

Proof. Since $\operatorname{Dom}\mathcal{K}^* \cap \operatorname{Dom}\mathcal{K}$ is a dense subset of $\mathcal{L}^2$, one can find $\{h_i,\, i \geq 1\}$ an ONB of $\mathcal{L}^2$ where for any i, h_i belongs to $\operatorname{Dom}\mathcal{K}^* \cap \operatorname{Dom}\mathcal{K}$. Set π_n the orthogonal projection in $\mathcal{L}^2$ onto the vector space spanned by $h_1, \ldots, h_n$. Let $V_k = \sigma\{\delta h_i, i = 1, \ldots, k\}$ and consider $u_{k,n} = \pi_n \mathrm{E}\left[P_{1/k} u \,|\, V_k\right]$ where P_t denote the Ornstein-Uhlenbeck semi-group of the Wiener process X. It is known, see [UZ00, Lemma B.6.1], that u_k can be written as

$$u_{k,n} = \sum_{i=1}^n f_i^n(\delta h_1, \ldots, \delta h_k) h_i \text{ where } f_i \in \mathcal{C}^\infty \text{ for any } i,$$

and that $u_{k,n}$ converges to u in $\mathbb{D}_{2,1}$. Furthermore, it is clear that we have

$$\begin{aligned}\operatorname{trace}((\mathcal{K}\nabla)u_{k,n}) &= \operatorname{trace}\sum_{i,j}\partial_j f_i^n(\delta h_1,\dots,\delta h_k)h_i\otimes\mathcal{K}h_j \\ &= \sum_{i,j}\partial_j f_i^n(\delta h_1,\dots,\delta h_k)\int_0^1 h_i(s)(\mathcal{K}h_j)(s)\,ds \\ &= \sum_{i,j}\partial_j f_i^n(\delta h_1,\dots,\delta h_k)\int_0^1 (\mathcal{K}^*h_i)(s)h_j(s)\,ds \\ &= \operatorname{trace}(\nabla(\mathcal{K}^*u_{k,n})). \end{aligned} \tag{39}$$

Moreover, if $\operatorname{trace}((\mathcal{K}\nabla)u)$ exists a.s., then the series

$$\sum_i < (\mathcal{K}\nabla)u,\, h_i\otimes h_i >_{\mathcal{L}^2\otimes\mathcal{L}^2} \text{ is convergent.}$$

Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\operatorname{trace}((\mathcal{K}\nabla)u_{k,n}) - \operatorname{trace}((\mathcal{K}\nabla)u)\right| \\ &\le \sum_{i\le n} < (\mathcal{K}\nabla)u_{k,n} - (\mathcal{K}\nabla)u, h_i\otimes h_i >_{\mathcal{L}^2\otimes\mathcal{L}^2} + \sum_{i>n} | < (\mathcal{K}\nabla)u,\, h_i\otimes h_i >_{\mathcal{L}^2\otimes\mathcal{L}^2} | \\ &\qquad \le n.\|(\mathcal{K}\nabla)(u-u_{k,n})\|_{\mathcal{L}^2\otimes\mathcal{L}^2} + \sum_{i>n} | < (\mathcal{K}\nabla)u,\, h_i\otimes h_i >_{\mathcal{L}^2\otimes\mathcal{L}^2} |.\end{aligned}$$

As n goes to infinity, the rightmost term converges a.s. to 0, hence for $\varepsilon > 0$, one can find n such that

$$\mathrm{P}(\sum_{i>n} | < (\mathcal{K}\nabla)u,\, h_i\otimes h_i >_{\mathcal{L}^2\otimes\mathcal{L}^2} > \varepsilon/2) \le \varepsilon/2.$$

Since $\mathcal{K}$ is a closed map, for this value of n, one can find k_n such that

$$\mathrm{P}(\|(\mathcal{K}\nabla)(u-u_{k_n,n})\|_{\mathcal{L}^2\otimes\mathcal{L}^2} > \varepsilon/2n) \le \varepsilon/2.$$

For such n and k_n, we have

$$\mathrm{P}(\left|\operatorname{trace}((\mathcal{K}\nabla)u_{k_n,n}) - \operatorname{trace}((\mathcal{K}\nabla)u)\right| > \varepsilon) \le \varepsilon.$$

Hence there exists a subsequence (k_j,n_j) such that $\operatorname{trace}((\mathcal{K}\nabla)u_{k_j,n_j})$ converges P-almost surely, thus that $\operatorname{trace}(\nabla(\mathcal{K}^*u))$ is finite and that the two expressions are equal. $\operatorname{trace}(\nabla(\mathcal{K}^*u)) = \operatorname{trace}((\mathcal{K}\nabla)u)$.

The very same reasoning holds true when $\operatorname{trace}(\nabla(\mathcal{K}^*u))$ is finite. □

Following [Nua95], we know that when u belongs to the domain of the two integrals (that of definition 4.1 and that of the last definition), these two integrals coincide.

A nice feature of this version of the stochastic integral is that we can compute its transformation under absolutely continuous change of probability.

Theorem 7.1. *Let $T(\omega) = \omega + Kv(\omega)$ be such that v belongs to $\mathbb{D}_{p,1}(\mathcal{L}^2)$ for some $p > 1$ and $T^*P \ll P$. Let u be such that u and $u \circ T$ belong to $\operatorname{Dom}\delta_{\mathcal{K}^*}$ and $\nabla\mathcal{K}^*u$ and $\nabla(\mathcal{K}^*u \circ T)$ are a.s. trace class operators. Then,*

$$\left(\int_0^1 u(s) * \, dX_s\right) \circ T = \int_0^1 (u \circ T)(s) * \, dX_s + \int_0^1 \mathcal{K}^*(u \circ T)(s)v(s)\,ds.$$

Proof. Theorem B.6.12 of [UZ00] stands that

$$\delta(\mathcal{K}^*u) \circ T = \delta(\mathcal{K}^*(u \circ T)) + \int \mathcal{K}^*(u \circ T)(s)v(s)\,ds + \operatorname{trace}((\nabla\mathcal{K}^*u) \circ T.\nabla v).$$

Proposition B.6.8 of [UZ00] implies that

$$\operatorname{trace}((\nabla\mathcal{K}^*u) \circ T.\nabla v) = \operatorname{trace}(\nabla(\mathcal{K}^*u \circ T)) - \operatorname{trace}(\nabla\mathcal{K}^*u) \circ T.$$

The proof is completed by substituting the latter equation into the former. □

For u deterministic and v adapted, this means that the law of the process $\{\int_0^t u_s dX_s - \int_0^t \mathcal{K}^*u(s)v(s)\,ds, t \geq 0\}$, under T^*P, is identical to the P-law of the process $\{\int_0^t u_s dX_s, t \geq 0\}$.

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