Optimal Space-Time Codes for the Amplify-and-Forward Cooperative Channel

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Abstract
In this work, we construct a family of space-time block codes for a half-duplex non-orthogonal Amplify-and-Forward fading cooperative channel with $N$ relays. We show that our construction achieves the optimal Diversity-Multiplexing tradeoff with a block length of $4N$.

Notation
Boldface lower case letters $v$ denote vectors. Boldface capital letters $M$ denote matrices. $CN$ represents the complex Gaussian random variable. $[\cdot]^T, [\cdot]^\dagger$ denote the matrix transposition and conjugated transposition operations. $|S|$ stands for the cardinality of the set $S$. $(x)^+$ means max$(0, x)$. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ and $\mathbb{Z}$ stand for the real field, complex field, rational field and the integer ring respectively. For each algebraic number field $K$, the ring of integers is denoted $\mathcal{O}_K$. For sake of simplicity, the “dot operators” are used all through the document to denote the high SNR behavior of different quantities. More precisely,

- for any quantity $q$,
  \[ q = \text{SNR}^\alpha \quad \text{means} \quad \lim_{\text{SNR} \to \infty} \frac{\log q}{\log \text{SNR}} = \alpha \]
  and similarly for $\lesssim$ and $\gtrsim$;
- for sets,
  \[ S_1 \doteq S_2 \quad \text{means} \quad \text{Prob} \{ s \in S_1 \} \doteq \text{Prob} \{ s \in S_2 \} \]

1 Introduction
In a wireless channel, diversity techniques are used to combat channel fadings. Recently, there has been a growing interest in the so called cooperative diversity techniques, where multiple terminals in a network cooperate to form a virtual antenna array in order to exploit spatial diversity in a distributed fashion. In this manner, diversity gain can be obtained even when local antenna array is not available. Since the work of [1,2], several cooperative diversity schemes have been proposed [3–6]. These schemes can be categorized into two principal classes: Amplify-and-Forward (AF) and Decode-and-Forward (DF).
In practice, the AF scheme is more attractive for its low complexity since the cooperative terminals (relays) simply forward the signal and do not decode it. In [6], Azarian et al. show that the non-orthogonal AF (NAF) scheme outperforms all previously proposed AF schemes in terms of the fundamental Diversity-Multiplexing (D-M) tradeoff [7]. The superiority of the NAF scheme comes from the fact that the source is allowed to transmit all the time, which boosts up the multiplexing gain. However, even though it is shown in [6] that the D-M tradeoff of this scheme can be achieved using a Gaussian random code of sufficiently large block length, no practical coding scheme that achieves the tradeoff has been proposed since then.

The main contribution of our work is an explicit algebraic construction of short block codes that achieve the D-M tradeoff for the multi-relay NAF scheme. Our construction is inspired by the non-vanishing determinant (NVD) space-time codes design for MIMO Rayleigh channels [8]. First, we show that for any linear fading channel, in the high SNR regime, the error event of a “good” space-time code \( \mathcal{X} \) (which will be properly defined later) occurs only when the channel is in outage. Therefore the optimal D-M tradeoff can always be achieved by \( \mathcal{X} \). Then, we will derive design criteria for the optimal codes that achieve the D-M tradeoff of the NAF scheme. Finally, an explicit algebraic construction is also shown, followed by some examples. The performance of our construction is confirmed by simulation results.

2 System Model and Problem Formulation

2.1 Preliminaries

For convenience of demonstration, we will first give out some definitions.

**Definition 1** (Exponential order [6]). For any nonnegative random variable \( x \), the Exponential order is defined as

\[
\xi \triangleq - \lim_{\text{SNR} \to \infty} \frac{\log x}{\log \text{SNR}}
\]  

(1)

We can denote \( x \doteq \text{SNR}^{-\xi} \).

**Definition 2** (Multiplexing and diversity gain [7]). A coding scheme \( \{C(\text{SNR})\} \) is said to achieve multiplexing gain \( r \) and diversity gain \( d \) if

\[
\lim_{\text{SNR} \to \infty} \frac{R(\text{SNR})}{\log \text{SNR}} = r \quad \text{and} \quad \lim_{\text{SNR} \to \infty} \frac{P_e(\text{SNR})}{\log \text{SNR}} = -d
\]

where \( R(\text{SNR}) \) is the data rate measured by bits per channel use (PCU) and \( P_e(\text{SNR}) \) is the average error probability using the maximum likelihood (ML) decoder.

**Definition 3** (rate-\( n \) NVD code). Let \( \mathcal{A} \) be an alphabet that is scalably dense, i.e.,

\[
|\mathcal{A}(\text{SNR})| \doteq \text{SNR}^{\frac{r}{n}}
\]

and \( a \in \mathcal{A}(\text{SNR}) \Rightarrow |a|^2 \leq \text{SNR}^{\frac{r}{n}} \)

for \( 0 \leq r \leq n \). Then, an \( n_t \times n_t \) space-time code \( \mathcal{X} \) is called a rate-\( n \) NVD code if it

1. is \( \mathcal{A} \)-linear\(^1\)

2. transmits on average \( n \) symbols PCU from the signal constellation \( \mathcal{A} \).

3. has the non-vanishing determinant (NVD) property\(^2\).

\(^1\)It means that every entry of any codeword \( X \in \mathcal{X} \) is a linear combination of symbols from \( \mathcal{A} \).

\(^2\)NVD means that \( |\det(X_i - X_j)| \geq \kappa > 0 \), \( \forall X_i, X_j \in \mathcal{X}, X_i \neq X_j \) with \( \kappa \) independent of the SNR.
2.2 Channel Model

We consider a network of \( N + 1 \) sources (users) and one destination. All the terminals are equipped with one antenna. The channel is shared in a TDMA manner, \( i.e., \) each user is allocated a time slot for the transmission of its own data. Within the same time slot, any of the other \( N \) users can help the current user transmit its own information. Suppose that the network configuration is symmetric. Without loss of generality, we consider only one time slot and the channel model becomes a single-user relay channel with one source, \( N \) relays and one destination, as shown in Fig. 1. The variables \( f \), \( h_i \)’s and \( g_i \)’s stand for the channel coefficients that remain constant during a block of length \( L \). As in the previous cited works, we assume that all the terminals work in half duplex mode, \( i.e., \) they cannot receive and transmit at the same time. The channel state information (CSI) is supposed to be known to the receiver but not to the transmitter.

2.3 The Amplify-and-Forward Relay Channel

In our work, we consider the non-orthogonal amplify-and-forward (NAF) scheme proposed in [5, 6]. In this scheme, the relays simply scale and forward the signal they receive.

\[
\begin{align*}
\text{source} & \quad x_1^T & \quad x_2^T \\
\text{relay} & \quad y_r & \quad by_r^T \\
\text{destination} & \quad y_1 & \quad y_2 \\
0 & \quad l/2 & \quad l
\end{align*}
\]

Figure 2: The NAF frame structure of a single-relay channel

In the single-relay case, each frame is composed of two partitions of \( l/2 \) symbols\(^3\). The frame length \( l \) is supposed to be smaller than the channel coherent time \( L \), \( i.e., \) the channel is static during the transmission of a frame. The half duplex constraint imposes that the relay can only transmit in the second partition. The frame structure is illustrated in Fig. 2, where the signal model is

\[
\begin{align*}
\begin{cases}
y_1 &= \sqrt{\pi_1 \text{SNR}} f x_1 + v_1 \\
y_r &= \sqrt{\pi_1 \rho \text{SNR}} h x_1 + w \\
y_2 &= \sqrt{\pi_3 \text{SNR}} g(by_r) + \sqrt{\pi_2 \text{SNR}} f x_2 + v_2
\end{cases}
\end{align*}
\]

\[^3\]It is shown in [6] that giving same length to the two partitions is optimal in terms of D-M tradeoff.
The channel is subject to a short-term power constraint, i.e., the power allocation factors $\pi_i$’s do not depend on the instantaneous channel realization $f, g$ and $h$, but can depend on $\rho$ and SNR. We impose that $\sum_i \pi_i = 2$ so that SNR represents the average received SNR at the destination.

As shown in [6], (2) is equivalent to $l/2$ channel uses of a $2 \times 2$ channel $\tilde{H}$ by writing

$$\tilde{y}_i = \sqrt{\text{SNR}} \tilde{H} \tilde{x}_i + z_i \quad \text{for } i = 1 \ldots l/2$$

where $\tilde{\xi}_i = [\xi_{1}[i] \xi_{2}[i]]^T$ with $\xi \in \{x, y\}$ and $\xi_{k}[i]$ denoting the $i$th symbol in the $k$th partition, $z_i \sim \mathcal{CN}(0, I)$ is the equivalent AWGN and

$$\tilde{H} \triangleq \begin{bmatrix} \frac{\sqrt{\pi_1 f}}{1 + \pi_3 \text{SNR} |bg|^2} & 0 \\ \frac{\pi_2}{\sqrt{1 + \pi_3 \text{SNR} |bg|^2}} & \frac{\pi_3}{\sqrt{1 + \pi_3 \text{SNR} |bg|^2}} \end{bmatrix}. \quad (3)$$

In the $N$–relay case, a super-frame of $N$ consecutive cooperation frames is defined, as shown in Fig. 3. It is assumed that the channel is static during the transmission of the super-frame (of $N \cdot l$ symbols). Within each frame, the source cooperates with only one relay. The cooperation is exactly the same as in the single-relay case. However, by encoding over the whole super-frame, a diversity order of $N + 1$ can be obtained.

The optimal D-M tradeoff achieved by the NAF protocol with $N$ relays is [6]

$$d_{\text{NAF},N}(r) = (1 - r)^{+} + N(1 - 2r)^{+} \quad (4)$$

where the achievability of the D-M tradeoff is proved by using a Gaussian random codebook with sufficiently long block length.

### 3 Optimal Codes Design Criteria

In this section, we derive design criteria for a family of short codes that achieve the optimal D-M tradeoff (4).
3.1 A General Result

The following theorem is fundamental to our construction.

**Theorem 1.** For any linear block fading channel

\[ Y = \sqrt{\text{SNR}} \, H \, x + z \]

where \( H \) is an \( n_r \times n_t \) matrix with \( q \leq \min\{n_r, n_t\} \) and \( z \) is the AWGN with i.i.d. entries, the achievable D-M tradeoff of a rate-\( n \) NVD code \( \mathcal{X} \) satisfies

\[ d_x(r) \geq d_{\text{out}} \left( \frac{q}{n} r \right) \]  

where \( d_{\text{out}}(r) \) is the outage upper bound of the D-M tradeoff for the channel \( H \).

**Proof.** See Appendix A.

In particular, for a full rate code \((n = q)\), the upper bound \( d_{\text{out}}(r) \) is achievable. This theorem implies that the NVD property is fundamental for \( \mathcal{X} \) to achieve all the diversity gain \( d \), for any linear fading channel. For a given diversity gain \( d \), the achievable multiplexing gain \( r \) of such \( \mathcal{X} \) is a shrunk version of \( r_{\text{out}}(d) \), the best one we can have for channel \( H \). Note that another such general result, concerning the full rate case, has been derived independently in [9]. But Theorem 1 is more adapted to the algebraic construction of explicit codes for the relay channel.

3.2 Design Criteria

With Theorem 1, we are ready to give out the design criteria for the construction of optimal codes for the NAF relay channel. The following theorem states the main result of our work.

**Theorem 2.** Let \( \mathcal{X} \) be a rate-2 NVD block diagonal code, i.e.,

\[ \mathcal{X} = \text{diag}(\Xi_1, \ldots, \Xi_N), \quad \forall \mathcal{X} \in \mathcal{X} \]  

where \( \Xi_i \)'s are \( 2 \times 2 \) matrices. Now consider a code \( \mathcal{C} \) with codewords in the form

\[ c^T = [c_1^T \ldots c_N^T] \]

with

\[ c_i = [\Xi_i(1, 1) \, \Xi_i(1, 2) \, \Xi_i(2, 1) \, \Xi_i(2, 2)]^T. \]

Then \( \mathcal{C} \) achieves the optimal D-M tradeoff \((4)\) of the NAF \( N \)-relay channel, by transmitting \( c_i^T \) in the \( i \)th cooperation frame. The code length of \( \mathcal{C} \) is \( 4N \).

**Sketch of proof.** Consider the \( N \)-relay NAF channel as a block diagonal channel \( H \). By construction, the D-M tradeoff of the code \((6)\) satisfies

\[ d_x(r) \geq d_{\text{out}}^\text{naf}(Nr) \]

according to Theorem 1. Since one channel use of \( H \) corresponds to a transmission of a NAF super-frame, we have

\[ d_c(r) = d_x(2r) \geq d_{\text{out}}^\text{naf}(2Nr) = d_{\text{NAF},N}(r). \]
4 A Unified Construction Framework

In this section, we assume that the modulation used by the source is a QAM. The field representing the modulated symbols is $\mathbb{Q}(i)$.

4.1 Codes Construction

We use the same methods as in [10]. The main difference is in the choice of the base field $\mathbb{F}$. In [10], this base field was equal to $\mathbb{Q}(i)$ or $\mathbb{Q}(j)$. Here, we choose as $\mathbb{F}$ a Galois extension of $\mathbb{Q}(i)$ with degree $N$ and denote $\tau_i$, $i = 1, \ldots, N$ the elements of its Galois group $\text{Gal}_{\mathbb{F}/\mathbb{Q}(i)}$. Now, we construct a cyclic algebra whose center is $\mathbb{F}$. We need a cyclic extension over $\mathbb{F}$ of degree 2. We denote it $\mathbb{K}$. The generator of its Galois group is $\sigma$.

The code construction needs two steps.

1. Construction of the cyclic algebra

   \[ \mathcal{A} = \{ z_0 + z_1 \cdot e \mid z_i \in \mathbb{K} \} \]  \hspace{1cm} (7)

   such that $e^2 = \gamma \in \mathbb{F}$ and $z_i \cdot e = e \cdot \sigma(z_i)$. In the matrix representation, we have

   \[ \mathcal{M}(e) = \begin{bmatrix} 0 & 1 \\ \gamma & 0 \end{bmatrix} \]

   and $\mathcal{M}(z_i) = \text{diag}(z_i, \sigma(z_i))$.


In terms of matrices, we construct, in step 1, the square $2 \times 2$ matrices $\Xi$. Then, by applying the canonical embedding of $\mathbb{F}$, the codeword is

\[ X = \text{diag}(\tau_1(\Xi), \ldots, \tau_N(\Xi)) \]  \hspace{1cm} (8)

where we can identify $\tau_i(\Xi) = \Xi_i$ from (6). As usual, we restrict the information symbols to be in $\mathbb{Z}[i]$. The infinite space-time code is defined as being the set of all matrices

\[ \mathcal{C} = \left\{ X = \begin{bmatrix} \mathcal{M}(\tau_1(z_0 + z_1e)) \cdots 0 \\ \vdots \ddots \vdots \\ 0 \cdots \mathcal{M}(\tau_N(z_0 + z_1e)) \end{bmatrix}, z_i \in \mathcal{O}_\mathbb{K} \right\} \]  \hspace{1cm} (9)

4.2 Codes Properties

The following lemma is obvious by construction.

**Lemma 1.** The code $\mathcal{C}$ of (9) is full rate.

**Lemma 2.** The code $\mathcal{C}$ is full rank iff

\[ \gamma \notin \mathbb{N}_{\mathbb{K}/\mathbb{F}}(\mathbb{K}) \]  \hspace{1cm} (10)

**Proof.** In [12], it is proved that a cyclic algebra is a division algebra (each element has an inverse) iff $\gamma$ is not a norm. \qed
Lemma 3. If 
\[ \gamma \notin N_{K/F}(\mathbb{K}) \]  
then, the code \( C \) has a non vanishing determinant, more precisely

\[ \delta_{\min} = \min_{\mathbf{X} \in C} \{ \det \mathbf{X} |^2 \in \mathbb{Z}^+ \setminus \{0\} \geq 1 \} \]

Proof. Because of the structure of \( \mathbf{X} \), its determinant is

\[ \det \mathbf{X} = \prod_{i=1}^{N} \det \tau_i(\Xi) = \prod_{i=1}^{N} \tau_i(\det(\Xi)) \]

But, \( \det(\Xi) \) is the reduced norm of \( z_0 + z_1 e \) thus it belongs to \( \mathcal{O}_F \). So,

\[ \prod_{i=1}^{N} \tau_i(\det(\Xi)) = N_{F/Q(i)}(\det(\Xi)) \in \mathbb{Z}[i]. \]

Since \( \det(\mathbf{X}) \neq 0 \) unless \( \mathbf{X} = 0 \), then we get \( \delta_{\min} \geq 1 \).

Finally, we derive the following result,

Theorem 3. The code \( C \) of (9) with \( z_i \in \mathcal{O}_K \) or a subgroup of \( \mathcal{O}_K \) (which will be in the following an ideal of \( \mathcal{O}_K \)) achieves the D-M tradeoff of the NAF \( N \)-relay channel.

Proof. The proof is straightforward and uses the results of the 3 above lemmas.

4.3 Shaping

As in [10, 13], we may be interested to construct codes that achieve the D-M tradeoff and that behave well in terms of error probability even for small alphabets such as QPSK (4-QAM). In that case, we add another constraint to our codes design, the shaping factor. This new constraint implies that \( |\gamma| = 1 \). Moreover, as in [10, 13], the linear transform that sends the vector composed by the \( 4N \) QAM information symbols to \( \text{vec}(\mathbf{X}) \) has to be unitary. The following examples will illustrate this claim.

5 Some Examples

5.1 The Single-Relay Channel

For the single-relay channel, the codewords are \( 2 \times 2 \) matrices. Because the Golden code satisfies to all the criteria of last section, then it achieves the D-M tradeoff for the single-relay channel.

5.2 The 2-Relay Channel

Optimal codes for the case \( N > 1 \) relays cannot be found in the literature. For the 2-relay case, we propose the following code refered to as \( C_{2,1} \). Codewords are block diagonal matrices with 2 blocks. Let \( \mathbb{F} = \mathbb{Q}(\zeta_8) \) with \( \zeta_8 = e^{\frac{2\pi i}{8}} \) be an extension of \( \mathbb{Q}(i) \) of degree 2. We choose \( \mathbb{K} = \mathbb{F}(\sqrt{5}) = \mathbb{Q}(\zeta_8, \sqrt{5}) \). In fact, we try to construct the Golden code on the base field \( \mathbb{Q}(\zeta_8) \) instead of the base field \( \mathbb{Q}(i) \). Moreover, the number \( \gamma \) is no more
equal to $i$ because $i$ is a norm in $\mathbb{Q}(\zeta_8)$ (in fact, $i = N_{\mathbb{K}/\mathbb{F}}(\zeta_8)$). We choose here, in order to preserve the shaping of the code, $\gamma = \zeta_8$. We prove that $\zeta_8 \notin N_{\mathbb{K}/\mathbb{F}}(\mathbb{K})$ [14] and thus that this code satisfies the full rank and non vanishing determinant conditions. Such a code uses 8 QAM symbols. Let $\theta = \frac{1+\sqrt{5}}{2}$, $\gamma = i$ and $\sigma : \theta \mapsto \bar{\theta} = \frac{1-\sqrt{5}}{2}$. The ring of integers of $\mathbb{K}$ is $\mathcal{O}_K = \{a + b\theta | a, b \in \mathbb{Z}[\zeta_8]\}$. Let $\alpha = 1 + i - i\bar{\theta}$ and $\bar{\alpha} = 1 + i - i\theta$. Codewords are given by

$$X = \begin{bmatrix} \Xi & 0 \\ 0 & \tau(\Xi) \end{bmatrix}$$

with

$$\Xi = \begin{bmatrix} \alpha \cdot (s_1 + s_2\zeta_8 + s_3\bar{\theta} + s_4\zeta_8\bar{\theta}) & \alpha \cdot (s_5 + s_6\zeta_8 + s_7\bar{\theta} + s_8\zeta_8\bar{\theta}) \\ \zeta_8\bar{\alpha} \cdot (s_5 + s_6\zeta_8 + s_7\bar{\theta} + s_8\zeta_8\bar{\theta}) & \bar{\alpha} \cdot (s_1 + s_2\zeta_8 + s_3\bar{\theta} + s_4\zeta_8\bar{\theta}) \end{bmatrix}$$

and $\tau$ maps $\zeta_8$ into $-\zeta_8$.

6 Numerical Results

In this section, we provide the simulation results on the performance of the codes proposed in Section 5. The performance is measured by the frame error rate (FER) vs received SNR per bit. For simplicity, we set the power allocation factors $\pi_1 = 2\pi_2 = 2\pi_3$. An optimization on the $\pi_i$'s as a function of $\rho$ and SNR can further improve the performance\footnote{A trivial suboptimal solution is to "turn on" the relay only when the $\rho$ and SNR are high enough to give a better performance over the no relay case}, but is not considered in this work. The transmitted signal constellation is 4-QAM. The geometric gain $\rho$ varies from 0 to 20 dB. Fig. 4 illustrates the performance of the Golden code on the single-relay channel. As a comparison, the performance of a non-cooperative channel is also shown in the figure. In this case, the frame length is 4 symbols. Compared to the non-cooperative case, the Golden code achieves diversity 2 and a gain of 12.5, 13.5, 14.3 and 14.8 dB is observed for $\rho = 0, 5, 10$ and 20 dB at FER $= 10^{-4}$. First of all, note that in the low SNR regime, the non-cooperative channel is better than the cooperative channel. In this regime, the error cumulation at the relay is more significant than the diversity gain provided by it. Then, we see that the difference between $\rho = 10$ dB and $\rho = 20$ dB is negligible. That is to say, a geometric gain of 10 dB is enough to achieve the (almost) best performance of the Golden code. In practice, it is often possible to find this kind of “helping agent” (with a geometric gain up to 10 dB). Same phenomena are observed for the 2-relay channel with the use of $C_{2,1}$ (Fig. 5). The frame length for the 2-relay is 8 symbols. For $\rho = 20$ dB, a gain of 19 dB at FER $= 10^{-4}$ is obtained.

7 Conclusion

We established, in this paper, a general framework to design short space-time block codes that achieve the diversity-multiplexing tradeoff of the amplify-and-forward cooperative channels, with an arbitrary number of relays. As examples, we gave explicit codes for one and two relays as well as simulation results.
Figure 4: 1-relay channel, 4-QAM

Figure 5: 2-relay channel, 4-QAM

A Proof of Theorem 1

We provide, here, a sketch of proof. A detailed version can be found in [14].

Let $\lambda_i$ denote the $i$th ordered eigenvalue of $HH^\dagger$ and $\alpha_i$ denote the exponential order of $\lambda_i$, i.e., $\lambda_i = \text{SNR}^{-\alpha_i}$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_q$. First, we show that for a channel $H$, the outage event at high SNR [7] is

$$O(r) \doteq \left\{ H : \log \det (I + \text{SNR}HH^\dagger) < r \log \text{SNR} \right\} = \left\{ \alpha : \sum_{k=j+1}^{q} \alpha_k \geq (n-j) - r, \ \forall j = 0, \ldots, q-1 \right\}$$

Then, we consider the error event of the rate-$n$ NVD code $X$. If we follow the footsteps of [8], by using the sphere bound, the average error probability of the code $X$ with ML decoding satisfies

$$P_e \leq \text{Prob}\{\eta \leq 0\}$$

with $-\eta$ being the exponential order of $d_{\text{min}}^2$, the minimum (squared) Euclidean distance between two different codewords in $X$. In addition, the properties of a rate-$n$ NVD code guarantee (by the mismatch eigenvalue bound [8]) that $d_{\text{min}}^2$ is lower bounded, with

$$d_{\text{min}}^2(\alpha) \doteq \text{SNR}^\eta \geq \text{SNR}^{\delta_j(\alpha)}, \quad j = 0, 1, \ldots, q-1$$

where

$$\delta_j(\alpha) = 1 - \frac{q}{n} \frac{r}{j+1} - \sum_{i=q-j}^{q} \alpha_i, \quad j = 0, 1, \ldots, q-1 \quad (12)$$

Finally, we get

$$P_e \leq \text{Prob}\{\eta \leq 0\} \leq \text{Prob}\{\delta_j \leq 0, \ \forall j = 0, 1, \ldots, q-1\} \leq \text{Prob}\left\{ \sum_{k=j+1}^{q} \alpha_k \geq (n-j) - \frac{q}{n} r, \ \forall j = 0, 1, \ldots, q-1 \right\} \doteq \text{Prob}\{\alpha \in O\left(\frac{q}{n} r\right)\}$$

which implies that

$$d_X(r) \geq d_{\text{out}}\left(\frac{q}{n} r\right).$$
References


