The invertibility of adapted perturbations of identity on the Wiener space

A. S. Üstünel and M. Zakai

Abstract: Let (W, H, μ) be the classical Wiener space. Assume that $U = I_W + u$ is an adapted perturbation of identity, i.e., $u : W \to H$ is adapted to the canonical filtration of W. We give some sufficient analytic conditions on u which imply the invertibility of the map U.

L'inversibilité des perturbations d'identité adaptées sur l'espace de Wiener

Resumé: Soit (W, H, μ) l'espace de Wiener. Soit $U = I_W + u$ une perturbation d'identité adaptée, i.e., $u : W \to H$ est adaptée à la filtration canonique de W. Nous donnons quelques conditions suffisantes qui impliquent l'inversibilité de l'application U.

1 Preliminaries

Let $W = C_0([0,1])$ be the Banach space of continuous functions on $[0,1]$, with its Borel sigma field denoted by $\mathcal{F}$. We denote by H the Cameron-Martin space, namely the space of absolutely continuous functions on $[0,1]$ with square integrable Lebesgue density:

$$H = \left\{ h \in W : \, h(t) = \int_0^t \dot{h}(s)ds, \, |h|_H^2 = \int_0^1 |\dot{h}(s)|^2 ds < \infty \right\} .$$

μ denotes the classical Wiener measure on $(W, \mathcal{F})$, $(\mathcal{F}_t, t \in [0,1])$ is the filtration generated by the paths of the Wiener process $(t, w) \to W_t(w)$, where $W_t(w)$ is defined as $w(t)$ for $w \in W$ and $t \in [0,1]$. We shall recall briefly some well-known functional analytic tools on the Wiener space, we refer the reader to [4, 3, 5] or to [6] for further details: $(P_\tau, \tau \in \mathbb{R}_+)$ denotes the semi-group of Ornstein-Uhlenbeck on W, defined as

$$P_\tau f(w) = \int_W f(e^{-\tau} w + \sqrt{1 - e^{-2\tau}} y) \mu(dy) \, .$$

Let us recall that $P_\tau = e^{-\tau \mathcal{L}}$, where $\mathcal{L}$ is the number operator. We denote by ∇ the Sobolev derivative which is the extension (with respect to the Wiener measure) of the Fréchet derivative in the Cameron-Martin space direction. The iterates of ∇ are defined similarly. Note that, if f is real valued, then ∇f is a vector and if u is an H-valued map, then ∇u is a Hilbert-Schmidt (on

H) operator valued map whenever defined. If Z is a separable Hilbert space and if $p > 1, k \in \mathbb{R}$, we denote by $\mathbb{D}_{p,k}(Z)$ the μ-equivalence classes of Z-valued measurable mappings ξ, defined on W such that $(I + \mathcal{L})^{k/2}\xi$ belongs to $L^p(\mu, Z)$ and this set, equipped with the norm

$$\|\xi\|_{p,k} = \|(I + \mathcal{L})^{k/2}\xi\|_{L^p(\mu,Z)} \tag{1.1}$$

becomes a Banach space. From the Meyer inequalities, we know that the norm defined by

$$\sum_{k=0}^{n} \|\nabla^k \xi\|_{L^p(\mu, Z\otimes H^{\otimes k})} \,,\; n \in \mathbb{N}\,,$$

is equivalent to the norm $\|\xi\|_{p,n}$ defined by (1.1). We denote by δ the adjoint of ∇ under μ and recall that, whenever $u \in \mathbb{D}_{p,0}(H)$ for some $p > 1$ is adapted, then δu is equal to the Itô integral of the Lebesgue density of u:

$$\delta u = \int_0^1 \dot{u}_s dW_s\,.$$

2 A sufficient condition for invertibility

Assume that $u : W \to H$ is adapted, i.e., $u(t) = \int_0^t \dot{u}_s ds$, $t \in [0, 1]$ and that $\dot{u}_s$ is $\mathcal{F}_s$-measurable for almost all $s \in [0, 1]$. We suppose that $\rho(-\delta u)$ defined as

$$\rho(-\delta u) = \exp\left[-\delta u - \frac{1}{2}|u|_H^2\right]$$

is the terminal value of a uniformly integrable martingale. We shall assume that u is in $\mathbb{D}_{2,0}(H)$. We have

Theorem 1 *Assume that u satisfies the hypothesis above. For $\tau \in [0, 1]$, define u_τ as to be $P_\tau u$, where P_τ is the Ornstein-Uhlenbeck semigroup and assume also that $E[\rho(-\delta u_\tau)] = 1$ for $\tau \in [0, 1]$. Then the adapted perturbation of identity $U = I_W + u$ is invertible provided that*

$$E\left[\int_0^1 |(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau|_H \rho(-\delta u_\tau) d\tau\right] < \infty\,. \tag{2.2}$$

Proof: Note that the map u_τ is again adapted and $H - C^1$ (in fact it is even $H - C^\infty$, cf. [7]). This means that there exists a negligeable set $N \subset W$ (in fact its capacity is null [6]) with $H + N \subset N$, such that, for any $w \in N^c$, the map $h \to u_\tau(w + h)$ is continuously Fréchet differentiable on H. Consequently $U_\tau = I_W + u_\tau$ satisfies the change of variables formula: for any $f \in C_b(W)$,

$$E[f \circ U_\tau\, \rho(-\delta u_\tau)] = E[f(w)\, N_\tau(w)]\,,$$

where N_τ is the multiplicity function of U_τ, namely the cardinality of the set $U_\tau^{-1}(\{w\})$(cf. [7]). Since $E[\rho(-\delta u_\tau)] = 1$, it follows that $N_\tau = 1$ μ-almost surely and this implies the existence of the inverse of U_τ which is denoted as V_τ. Note that V_τ is of the form $V_\tau = I_W + v_\tau$, where $v_\tau : W \to H$ and that the image of μ under V_τ, denoted as $V_\tau\mu$, is equivalent to μ with the Radon-Nikodym density

$$\frac{dV_\tau\mu}{d\mu} = \rho(-\delta u_\tau)\,. \tag{2.3}$$

We also have $v_\tau = -u_\tau \circ V_\tau$. We shall prove that $\lim_{\tau\to 0} v_\tau$ exists in $L^0(\mu, H)$. Note that $\tau \to v_\tau$ is differentiable on $(0,1)$ and we have

$$\frac{dv_\tau}{d\tau} = -\left((I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right) \circ V_\tau\,. \tag{2.4}$$

Since

$$|v_\beta - v_\alpha| \le \int_\alpha^\beta \left|\frac{dv_\tau}{d\tau}\right|_H d\tau\,,$$

and since $L^0(\mu, H)$ is complete, in order to show that $\lim_{\alpha,\beta\to 0} \mu\left(\{|v_\alpha - v_\beta| > c\}\right) = 0$, for any $c > 0$, it suffices to show that

$$E\int_0^\kappa \left|\frac{dv_\tau}{d\tau}\right| d\tau < \infty\,,$$

for some $\kappa > 0$. From the relations (2.3) and (2.4), we obtain

$$\begin{aligned} E\int_\alpha^\beta \left|\frac{dv_\tau}{d\tau}\right|_H d\tau &= E\int_\alpha^\beta \left|\left((I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau\right) \circ V_\tau\right|_H d\tau \\ &= E\int_\alpha^\beta |(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau|_H \rho(-\delta u_\tau) d\tau\,. \end{aligned}$$

Hence the hypothesis (2.2) implies the existence of the limit $\lim_{\tau\to 0} v_\tau$ in $L^1(\mu, H)$ which we shall denote by v. Since $v_\tau = -u_\tau \circ V_\tau$ and since $(\rho(-\delta u_\tau),\ \tau \in [0,1])$ is uniformly integrable, $V\mu$ is absolutely continuous with respect to μ and we have also the identity $v = -u \circ V$, where $V = I_W + v$. Now it is easy to see that $U \circ V = V \circ U = I_W$ μ-almost surely. □

Combining Theorem 1 with the inequality of T. Carleman, (cf. [1] or [2], Corollary XI.6.28) which says:

$$\|\det_2(I_H + A)(I_H + A)^{-1}\| \le \exp\frac{1}{2}\left(\|A\|_2^2 + 1\right)\,,$$

for any Hilbert-Schmidt operator A, where the left hand side is the operator norm, $\det_2(I_H + A)$ denotes the modified Carleman-Fredholm determinant and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, we get

Theorem 2 *Assume that $u \in \mathbb{D}_{2,1}(H)$ such that $E[\rho(-\delta u_\tau)] = 1$ and that*

$$E\left[e^{\frac{1}{2}\|\nabla u\|_2^2}\int_0^1 P_\tau\left(\rho(-\delta u_\tau)|\mathcal{L}u_\tau|_H\right) d\tau\right] < \infty\,.$$

Then U satisfies the conclusions of Theorem 1.

Proof: The integrand in the relation (2.2) can be upperbounded as follows:

$$\begin{aligned} |(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau|_H &\le \exp\frac{1}{2}(\|\nabla u_\tau\|_2^2 + 1)|\mathcal{L}u_\tau|_H \\ &\le |\mathcal{L}u_\tau|_H P_\tau(\exp\frac{1}{2}(\|\nabla u\|_2^2 + 1))\,, \end{aligned}$$

where the second line follows from the Jensen inequality. Here there is no term with $\det_2$ since, ∇u_τ being quasi-nilpotent, its Carleman-Fredholm determinant is always equal to one. We then use the symmetry of P_τ with respect to μ. □

Corollary 1 *Suppose that u is adapted, $E[\rho(-\delta u_\tau)] = 1$ for all $\tau \in [0,1]$. Let $\varepsilon > 0$ be given and assume further that $u \in \mathbb{D}_{\frac{\varepsilon+1}{\varepsilon},2}(H)$ and that the following relation holds:*

$$E\left[\left(1 + e^{-e(1+\varepsilon)\delta u}\right)\exp\left(\frac{1+\varepsilon}{2}\|\nabla u\|_2^2\right)\right] < \infty\,. \tag{2.5}$$

Then $U = I_W + u$ is μ-almost surely invertible.

Proof: Let C_ε represent the left hand side of the relation (2.5), then using the Hölder inequality we get

$$E\left[\int_0^1 |(I_H + \nabla u_\tau)^{-1}\mathcal{L}u_\tau|_H \rho(-\delta u_\tau) d\tau\right] \leq C_\varepsilon^{\frac{1}{1+\varepsilon}} \|u\|_{\frac{1+\varepsilon}{\varepsilon},2}\,.$$

Hence the conclusion follows. □

Remark: If we take $\varepsilon = 1$ in Corollary 1, then it is easy to see, using the Wiener chaos expansion for $E[|\mathcal{L}P_\tau u|_H^2]$ that

$$E\int_0^1 |\mathcal{L}P_\tau u|_H^2 d\tau \leq \|u\|_{2,1}^2\,.$$

Remark: In the case where u is not adapted, the condition (2.5) with $\varepsilon = 1$ is sufficient for the measure theoretic degree of the map U to be one as it is proven in Theorem 9.3.2 of [7].

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A. S. Üstünel
ENST, Paris
Dept. Infres, 46, rue Barrault
75013 Paris,
France
ustunel@enst.fr

M. Zakai,
Technion, Haifa
Dept. Electrical Eng.,
32000 Haifa
Israel
zakai@ee.technion.ac.il