Subgeometric rates of convergence of f-ergodic strong Markov processes

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Abstract

We provide a condition for f-ergodicity of strong Markov processes at a subgeometric rate. This condition is couched in terms of a supermartingale property for a functional of the Markov process. Equivalent formulations in terms of a drift inequality on the extended generator and on the resolvent kernel are given. Results related to (f,r)-regularity and to moderate deviation principle for integral (bounded) functional are also derived. Applications to specific processes are considered, including elliptic stochastic differential equation, Langevin diffusions, hypoelliptic stochastic damping Hamiltonian system and storage models.

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1 Introduction

In the present paper, we study the recurrence of continuous-time Markov processes. More precisely, we provide a criterion that yields a precise control of a subgeometric moment of the return-time to a test-set. The obtained result permits further quantitative analysis of characteristics such as the regularity of the process, the rate of convergence to the stationary state, and a moderate deviation principle.

The stability and ergodic theory of continuous-time Markov processes has a large literature which is mainly devoted to the geometric case (also referred to as the exponential case). Meyn and Tweedie developed stability concepts for continuous-time Markov processes as well as simple criteria for non-explosivity, non-evanescence, Harris-recurrence, positive Harris-recurrence, ergodicity and geometric ergodicity [21, 23, 24]. Of particular importance in actually applying these concepts is the existence of verifiable conditions. In the discrete-time context, development of Foster-Lyapunov type conditions on the transition kernel has provided such criteria (e.g. [22]). In the continuous-time context, Foster-Lyapunov inequalities applied to the generator of the process play the same role. These criteria were successfully applied to the study of the solution to stochastic differential equations (see [16] and more recently, [14] and references therein). Results relative to rates of convergence slower than geometric are not so well established. In [32, 20] (resp. [34]), polynomial and sub-exponential ergodicity of stochastic differential equations (resp. sub-exponential ergodicity of queuing models) are addressed, but these results are quite model-specific. Fort and Roberts [11] are, to our best knowledge, the first to study the subgeometric ergodicity of general strong Markov processes. Their conditions are in terms of subgeometric moment of the return-time to a test-set. Fort and Roberts derive nested drift inequalities on the generator of the process that makes the result of practical interest in the polynomial case.

One of the application of the condition we derive in the present paper makes the Fort-Roberts's theory applicable for more general subgeometric rates such as the logarithmic or the subexponential ones. It also provides criteria for the (f, r)-regularity of a process, a characteristic which is an extension of the regularity concept [23]. We obtain theoretical results that are analogous to those in the discrete-time case [31]. We then relate our condition to a criterion based on the generator of the process. This criterion is the natural analogue of the Foster-Lyapunov condition for the geometric case; it also provides a single drift condition that generates the set of nested drift conditions by Fort-Roberts [11] for the polynomial case. Furthermore, it is analogous to the discrete-time version recently proposed by Douc-Fort-Moulines-Soulier [4]. In the literature, one approach for the theory of continuous-time Markov process is through the use of associated discrete-time chains : the resolvent chains and/or a skeleton chain. We discuss how our condition is related to a subgeometric drift inequality for these discrete-time Markov chains. As a consequence, we state new limit theorems such as moderate deviations for integral of bounded functionals, thus weakening the conditions derived in Guillin-Wu [15, 33].

Our conditions are then successfully applied to various non trivial models: *(a)* we first con-

sider elliptic stochastic differential equations for which conditions on the drift function enable us to generalize results by Veretennikov [32], Ganidis-Roynette-Simonot [12] or Malyshkin [20] (see also Pardoux-Veretennikov [27] for a study of the regularity of the solution of the Poisson equation under this drift condition); *(b)* we then study a "cold" Langevin tempered diffusion when the invariant target distribution is subexponential in the tails. This model is particularly useful in Markov Chain Monte Carlo method. Different regime of ergodicity (polynomial, subexponential or exponential) depending on the coldness of the diffusion term are exhibited, the different regimes are then characterized by the invariant target distribution. This study generalizes the Fort-Roberts' results, which consider the case when the target density is polynomial in the tails [11]; *(c)* we also give a toy hypoelliptic example, namely a stochastic damping Hamiltonian system, in the case when it cannot be geometrically ergodic. This model is shown to be polynomially ergodic (see Wu [33] for the exponential case); *(d)* we finally consider a simple compound Poisson-process driven Ornstein-Uhlenbeck process (relevant for recent studies in financial econometrics) with heavy tailed jump. It is shown to be subgeometrically ergodic.

Our approach may be considered as a probabilistic one. There are another ways to get subexponential rates of convergence (in total variation norm, in L^2 or in entropy) such as those based on spectral techniques (as in [12]), or on functional inequalities (weak Poincaré inequalities [29] or weak logarithmic Sobolev inequalities [1]). These results are however not easy to compare to ours and we postpone a comparative utilization of these approaches to further research.

Let us finally present the organization of the paper. Section 2 recalls basic definitions on Markov processes. The main results are given in Section 3. All the proofs are postponed in appendix. Section 4 is devoted to the examples and Section 5 to a new moderate deviation principle.

2   Definitions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathsf{X}})$ be a Markov family on a locally compact and separable metric space X endowed with its Borel σ-field $\mathcal{B}(\mathsf{X})$: $(\Omega, \mathcal{F})$ is a measurable space, $(X_t)_{t \geq 0}$ is a Markov process with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{P}_x$ (resp. $\mathbb{E}_x$) denotes the canonical probability (resp. expectation) associated to the Markov process with initial distribution the point mass at x. Throughout this paper, the process is assumed to be a time-homogeneous strong Markov process with cad-lag paths, and we denote by $(P_t)_{t \geq 0}$ the associated transition function on $(\mathsf{X}, \mathcal{B}(\mathsf{X}))$.

Let Λ_0 denote the class of the measurable and nondecreasing functions $r : [0, +\infty) \to [2, +\infty)$ such that $\log r(t)/t \downarrow 0$ as $t \to +\infty$. Let Λ denote the class of positive measurable functions $\bar{r}$, such that for some $r \in \Lambda_0$,

$$0 < \liminf_t \frac{\bar{r}(t)}{r(t)} \leq \limsup_t \frac{\bar{r}(t)}{r(t)} < \infty.$$

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Λ is the class of the subgeometric rate functions and examples of functions $\bar{r} \in \Lambda$ are

$$\bar{r}(t) = t^{\alpha} \, (\log t)^{\beta} \, \exp(\gamma t^{\delta})$$

for $0 < \delta < 1$ and either $\gamma > 0$, or $\gamma = 0$ and $\alpha > 0$, or $\gamma = \alpha = 0$ and $\beta \geq 0$. We are ultimately interested in conditions implying that for all $x \in \mathsf{X}$

$$\lim_{t \to +\infty} \, r(t) \, \|P^t(x, \cdot) - \pi(\cdot)\|_f = 0, \tag{2.1}$$

where $r \in \Lambda$, π is the (unique) invariant distribution of the process *i.e.* $\pi P^t = \pi$ for all $t \geq 0$, and for a signed measure μ, $\|\mu\|_f = \sup_{|g| \leq f} |\mu(g)|$ where $f : \mathsf{X} \to [1, \infty)$ is a measurable function. When f is the constant function $\mathbf{1}$ ($\mathbf{1}(t) = 1$, $t \geq 0$), the f-norm is nothing more than the total variation norm.

To that goal, we will need different notions of regularity and stability of continuous-time Markov processes and we briefly recall some basic definitions. The process is ϕ-irreducible for some σ-finite measure ϕ on $\mathcal{B}(\mathsf{X})$ if $\phi(A) > 0$ implies $\mathbb{E}_x \left[\int_0^\infty \mathbf{1}_A(X_s) \, ds \right] > 0$ for all $x \in \mathsf{X}$. A ϕ-irreducible process possesses a maximal irreducibility measure ψ such that ϕ is absolutely continuous with respect to ψ for any other irreducibility measure ϕ [25]. Maximal irreducibility measures are not unique and are equivalent. A set $A \in \mathcal{B}(\mathsf{X})$ such that $\psi(A) > 0$ for some maximal irreducibility measure is said accessible; and full if $\psi(A^c) = 0$. A measurable set C is ν_a-petite (or simply petite) if there exist a probability measure a on the Borel σ-field of $[0, +\infty)$ and a non-trivial σ-finite measure ν_a on $\mathcal{B}(\mathsf{X})$ such that

$$\forall x \in C, \qquad \int_0^{+\infty} P^t(x, \cdot) \, a(dt) \geq \nu_a(\cdot).$$

For a ψ-irreducible process, an accessible closed petite set always exists [21]. A process is Harris-recurrent if, for some σ-finite measure μ, $\mu(A) > 0$ implies that the event $\{ \int_0^\infty \mathbf{1}_A(X_s) ds = \infty \}$ holds $\mathbb{P}_x$-a.s. for all $x \in \mathsf{X}$. Harris-recurrence trivially implies ϕ-irreducibility. A Harris-recurrent right process possesses an invariant measure π [13]; if π is an invariant probability distribution, the process is called positive Harris-recurrent. A ϕ-irreducible process is aperiodic if there exists an accessible ν_{δ_m}-petite set C and t_0 such that for all $x \in C, t \geq t_0, P^t(x, C) > 0$. A sufficient condition for a positive Harris-recurrent process to be aperiodic is the existence of some ϕ-irreducible skeleton chain [23]; recall that a skeleton P^m ($m > 0$) is said ϕ-irreducible if there exists a σ-finite measure μ such that $\mu(A) > 0$ implies $\forall x \in \mathsf{X}, \exists k \in \mathbb{N}, P^{km}(x, A) > 0$ [22].

A ψ-irreducible and aperiodic Markov process that verifies (2.1) is said f-ergodic at a subgeometric rate (or simply f-ergodic when $r = \mathbf{1}$). When r is of the form $r(t) = \kappa^t$ for some $\kappa > 1$, the process is said f-ergodic at a geometric rate. In the literature, criteria for the stability of Markov processes, when stability is couched in terms of Harris-recurrence, positive Harris-recurrence, f-ergodicity, with in this latter case, a mention of the rate of convergence, are expressed in terms of hitting-times of some closed petite set. For any $\delta > 0$ and any closed set $C \in \mathcal{B}(\mathsf{X})$, let

$$\tau_C(\delta) = \inf\{t \geq \delta, X_t \in C\},$$

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be the hitting-time on C delayed by δ and define its (f, r)-modulated moment

$$G_C(x, f, r; \delta) = \mathbb{E}_x \left[\int_0^{\tau_C(\delta)} r(s) \, f(X_s) \, ds \right],$$

where $f : \mathsf{X} \to [1, \infty)$ is a measurable function and $r : [0, +\infty) \to (0, +\infty)$ is a rate function. When $f = \mathbf{1}$ (resp. $r = \mathbf{1}$), this moment is simply called the r-modulated (resp. f-modulated) moment. Following discrete-time usage [22, 31, 17], we call a measurable set C (f, r)-regular if

$$\sup_{x \in C} G_B(x, f, r; \delta) < \infty,$$

for all $\delta > 0$ and all accessible set B. Criteria for Harris-recurrence and positive Harris-recurrence can be found in [21, Theorems 1.1 and 1.2]; ergodicity and f-ergodicity are addressed in [23, Theorems 6.1 and 7.2]; criteria for geometric f-ergodicity at a geometric rate (resp. at a subgeometric rate) are provided by [6, Theorem 7.4] (resp. [11, Theorem 1]). A short review of these notions and results can be found in [11].

In many applications, these moments can not be explicitly calculated; a second set of criteria based on the extended generator were thus derived for some of the stability properties above. We postpone to Section 3.4 a review of the existing conditions.

3 Main results

Let us consider the following drift condition towards a closed petite set C.

$\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$: There exist a closed petite set C, a cad-lag function $V : \mathsf{X} \to [1, \infty)$, an increasing differentiable concave positive function $\phi : [1, \infty) \to (0, \infty)$ and a constant $b < \infty$ such that for any $s \geq 0$, $x \in \mathsf{X}$,

$$\mathbb{E}_x \left[V(X_s) \right] + \mathbb{E}_x \left[\int_0^s \phi \circ V(X_u) du \right] \leq V(x) + b \, \mathbb{E}_x \left[\int_0^s \mathbf{1}_C(X_u) du \right]. \qquad (3.1)$$

Note that (3.1) is equivalent to the condition that the functional

$$s \mapsto V(X_s) - V(X_0) + \int_0^s \phi \circ V(X_u) du - b \int_0^s \mathbf{1}_C(X_u) du$$

is, for all $x \in \mathsf{X}$, a $\mathbb{P}_x$-supermartingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

The main result of Section 3.1 is Theorem 3.1 that states that this drift condition allows the calculation of an upper bound for some r-modulated moment where $r \in \Lambda$, and for some f-modulated moment, $f \geq 1$. Using interpolating inequalities, we obtain (f, r)-modulated moments for a wide family of pairs (f, r). Section 3.2 is devoted to (f, r)-regularity : the main result of this section is Proposition 3.7 that identifies (f, r)-regular sets from the condition

$\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$. In Section 3.3, we show that the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ provides a simple sufficient condition for an aperiodic strong Markov process to be f-ergodic at a subgeometric rate : the main result is Theorem 3.10 that builds on the work by Fort and Roberts [11]. We provide in Section 3.4 a condition couched in terms of the extended generators that implies the drift inequality $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$. This condition generalizes the condition in [11, Proposition 6] that restricts to the polynomial case, and reveals of great interest in many applications. We present in Section 3.5 the interplay between a drift condition on the resolvent kernel and the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$.

All the proofs are given in Appendix A.

3.1 Modulated moments

We show that $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ is a simple condition that allows the control of f-modulated moments and r-modulated moments, for a general rate function $r \in \Lambda$, of the delayed hitting-time $\tau_C(\delta)$. Let

$$H_\phi(u) = \int_1^u \frac{ds}{\phi(s)}, \qquad u \geq 1 .$$

Theorem 3.1. *Assume* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$.

i) For all $x \in \mathsf{X}$ and $\delta > 0$,

$$\mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \phi \circ V(X_s) \, ds \right] \leq V(x) - 1 + b\delta .$$

ii) For all $x \in \mathsf{X}$ and $\delta > 0$,

$$\mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \phi \circ H_\phi^{-1}(s) \, ds \right] \leq V(x) - 1 + \frac{b}{\phi(1)} \int_0^\delta \phi \circ H_\phi^{-1}(s) ds .$$

The proof of Theorem 3.1 does not require C to be petite. Nevertheless, this petiteness property will be crucial in all the following results: we will see that this assumption allows the extension of the above controls to those of modulated moments $\tau_B(\delta)$ for any accessible set B. Theorem 3.1 gives the largest f-modulated and r-modulated moments of $\tau_C(\delta)$ that can be deduced from $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$. Interpolated (f, r_f)-modulated moments of $\tau_C(\delta)$ can easily be obtained for a wide family of functions $1 \leq f \leq f_*$ (and, equivalently, a wide family of rate functions $r(s) \leq r_*(s)$) where

$$f_* = \phi \circ V, \qquad r_*(s) = \phi \circ H_\phi^{-1}(s). \qquad (3.2)$$

To that goal, we follow the same lines as in [4] and [11] and introduce the pairs of Young's functions (H_1, H_2) that, by definition, satisfy the property

$$x \, y \leq H_1(x) + H_2(y), \qquad \forall x, y \geq 0, \qquad (3.3)$$

and are invertible (see e.g [19, Chapter 1]). Let $\mathcal{I}$ be the pairs of inverse Young's functions augmented with the pairs $(\mathrm{Id}, \mathbf{1})$ and $(\mathbf{1}, \mathrm{Id})$. Examples of pairs (H_1, H_2) are given in [4] and [11] while a general construction can be found in [19, Chapter 1]. Corollary 3.2 trivially results from Theorem 3.1 and Eq. (3.3).

Corollary 3.2. *Assume* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$. *For any pairs* $(\Psi_1, \Psi_2) \in \mathcal{I}$ *and all* $\delta > 0$,

$$\mathbb{E}_x \left[\int_0^{\tau_C(\delta)} \Psi_1\left(r_*(s)\right) \ \Psi_2\left(f_*(X_s)\right) \ ds \right] \leq 2(V(x) - 1) + b \int_0^\delta \left(1 + \frac{r_*(s)}{r_*(0)}\right) ds.$$

For two pairs (Ψ_1, Ψ_2) and (Ψ_1', Ψ_2') in $\mathcal{I}$, if $\Psi_1(x) \leq \Psi_1'(x)$ for all large x, then $\Psi_2(y) \geq \Psi_2'(y)$ for all large y [19, Theorem 1.2.1]. This shows that the rate $\Psi_1\left(r_*(\cdot)\right)$ and the function $\Psi_2\left(f_*(\cdot)\right)$ have to be balanced : the maximal rate function r_* is associated to the function f with minimal growth in the range $1 \leq f \leq f_*$, that is with $f = \mathbf{1}$; and the function with the largest rapidity of growth $f = f_*$ is associated to the minimal rate $r = \mathbf{1}$.

Theorem 3.1 and Corollary 3.2 thus provides a control of (f, r)-modulated moments; a simple condition for the rate r to be in the set Λ of the subgeometric rate functions is recalled in the following lemma [4, Lemmas 2.3 and 2.7]

Lemma 3.3. *If* $\lim_\infty \phi' = 0$, $r_* \in \Lambda$ *and for all inverse Young function* Ψ_1, $\Psi_1 \circ r_* \in \Lambda$.

Proposition 3.4. *Assume* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$. *Then the process is* ψ-*irreducible. If* $\sup_C V < \infty$,

(i) *the level sets* $\{V \leq n\}$ *are petite and the union of these level sets is full.*

(ii) *there exists a closed accessible petite set* B *such that* $\mathbf{D}(\mathbf{B}, \mathbf{V}, \phi, \mathbf{b})$ *holds and* $\sup_B V < \infty$.

As a consequence, when $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ holds and $\sup_C V < \infty$, we can assume without loss of generality that C is accessible.

3.2 (f, r)-regularity

Corollary 3.2 shows that the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ allows the control of modulated moments $G_C(x, f, r; \delta)$, for all $\delta > 0$ and a large family of pairs (f, r). Similar modulated moments relative to any accessible set B can be controlled provided $\sup_{x \in C} G_C(x, f, r; \delta) < \infty$ for some $\delta > 0$ (and thus any $\delta > 0$, as established in [11, Lemma 20]). This naturally yields the notion of (f, r)-regular sets. The objective of this section is to identify regular sets from the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$.

We start with a proposition that shows that the "self-regularity" of a closed petite set C actually implies (f, r)-regularity. This results extends [21, Proposition 4.1] (resp. [11, Proposition 22]) that addresses the case $r = \mathbf{1}$ (resp. $f = \mathbf{1}$). It also generalizes [11, Proposition 23] which concerns the case $r = \Psi_1(r_*)$ and $f = \Psi_2(f_*)$ for some pair $(\Psi_1, \Psi_2) \in \mathcal{I}$. This proposition is the counterpart in the subexponential setting of the result by Down-Meyn-Tweedie for the exponential case [6, Theorem 7.2].

Proposition 3.5. *Let $f : \mathsf{X} \to [1, \infty)$ be a measurable function and $r \in \Lambda$ be a subgeometric rate function. Assume that the process is ψ-irreducible and $\sup_{x \in C} G_C(x, f, r; \delta) < \infty$ for some (and thus any) $\delta > 0$ and some closed petite set C.*
For all accessible set $B \in \mathcal{B}(\mathsf{X})$ and all $t \geq 0$, there exists a constant $c_{B,t} < \infty$ such that for all $x \in \mathsf{X}$,

$$G_B(x, f, r; t) \leq c_{B,t} \, G_C(x, f, r; \delta). \tag{3.4}$$

Hence C is (f, r)-regular.

Proposition 3.6. *Assume that $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ holds with C, V, ϕ such that $\sup_C V < \infty$ and $\lim_{+\infty} \phi' = 0$. Then for any pair $(\Psi_1, \Psi_2) \in \mathcal{I}$, any accessible set B and all $\delta > 0$, there exists a finite constant c such that*

$$\mathbb{E}_x \left[\int_0^{\tau_B(\delta)} \Psi_1 \left(r_*(s) \right) \, \Psi_2 \left(f_*(X_s) \right) \, ds \right] \leq c \, V(x).$$

Hence, any V-level set $\{x \in \mathsf{X}, V(x) \leq v\}$ is (f, r)-regular for all pairs $(f, r) = (\Psi_2 \circ f_*, \Psi_1 \circ r_*)$ with $(\Psi_1, \Psi_2) \in \mathcal{I}$.

Proof. By Corollary 3.2, $\sup_{x \in C} G_C(x, f, r; \delta) < \infty$ for all $\delta > 0$ provided the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ holds and $\sup_C V < \infty$. Finally, $r = \Psi_1 \circ r_*$ for some inverse Young function Ψ_1 is a subgeometric rate if $\lim_{+\infty} \phi' = 0$. Proposition 3.5 thus implies that the level sets of V are (f, r)-regular sets. $\square$

We now establish a general result that extends to continuous-time Markov processes, part of [31, Theorem 2.1] relative to discrete-time Markov chain. In the case $r = \mathbf{1}$, some of these equivalences are proved in [21] for continuous-time strong Markov processes.

Proposition 3.7. *Let $f : \mathsf{X} \to [1, \infty)$ be a measurable function and $r \in \Lambda$ be a subgeometric rate function. Assume that the process is ψ-irreducible. The following conditions are equivalent*

  i) *There exist a closed petite set C and $\delta > 0$ such that $\sup_C G_C(x, f, r; \delta) < \infty$.*

 ii) *There exists a (f, r)-regular closed set which is accessible.*

iii) *There exists a full set $\mathcal{S}_\Psi$ which is the union of a countable number of (f, r)-regular sets.*

Theorem 3.1 proves that these equivalent conditions are verified provided $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ holds, $\sup_C V < \infty$ and $\lim_\infty \phi' = 0$.
We conclude this section by establishing that under mild additional conditions, the drift condition $\mathbf{D}$ also yields controls of modulated moments for the skeleton chains. For all $m > 0$, let $T_{m,C}$ be the return-time to C of the skeleton chain P^m,

$$T_{m,C} = \inf\{k \geq 1, X_{mk} \in C\}.$$

Proposition 3.8. *Assume that* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ *holds with* $\sup_C V < \infty$, *and some skeleton chain is irreducible. For all* $m > 0$ *and any accessible set* B, *there exist constants* $c_i, 1 \leq i \leq 4$, *such that for all* $x \in \mathsf{X}$,

$$\mathbb{E}_x \left[\sum_{k=0}^{T_{m,B}-1} \phi \circ V(X_{mk}) \right] \leq c_1 \, \mathbb{E}_x \left[\int_0^{T_{m,B}} \phi \circ V(X_{sm}) \, ds \right] \leq c_2 \, V(x),$$

and

$$\forall x \in \mathsf{X}, \quad \mathbb{E}_x \left[\sum_{k=0}^{T_{m,B}-1} r_*(km) \right] \leq c_3 \, \mathbb{E}_x \left[\int_0^{\tau_B(\delta)} r_*(s) \, ds \right] \leq c_4 \, V(x).$$

We will see in the last section that this proposition which clearly links the behavior of the skeleton chain to that of the initial process leads to new limit theorems such as moderate deviations. It will also imply interesting applications to averaging principle.

3.3 f-ergodicity at a subgeometric rate

From the control of $x \mapsto G_C(x, f, r; \delta)$ where C is a closed petite set, we are able to deduce results on the ergodic behavior of the strong Markov process.

The first result concerns the existence of an invariant probability distribution π and shows that the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ provides a simple tool when identifying the set of the π-integrable functions. The second one states that the Markov process converges in f-norm to the invariant probability measure π, for a wide family of functions $1 \leq f \leq f_*$ and a wide family of rate functions $r_f \leq r_*$.

Proposition 3.9. *Assume* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ *and* $\sup_C V < \infty$. *Then the process is positive Harris-recurrent with an invariant probability measure* π *such that* $\pi(\phi \circ V) < \infty$.

Proposition 3.9 results from [21, Theorems 1.1 and 1.2] and Theorem 3.1(i). It is known that positive Harris-recurrence does not necessarily imply ergodicity and aperiodicity is required [23, Proposition 6.1]; similar conditions are required in the discrete-time case [22]. In the present case, we have more information than positive Harris-recurrence and thus, we are able to establish f-ergodicity at a subgeometric rate.

For a sequence $r \in \Lambda$, define $r^0(t) = \int_0^t r(s) \, ds$, and, for a differentiable rate function r, set $\partial r(t) = \frac{dr(t)}{dt}$.

Theorem 3.10. *Assume that*

(i) *some skeleton chain is irreducible.*

(ii) *the condition* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ *holds with* C, V, ϕ *such that* $\sup_C V < \infty$ *and* $\lim_{+\infty} \phi' = 0$.

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For any pair $\Psi = (\Psi_1, \Psi_2) \in \mathcal{I}$ and any probability distribution λ satisfying $\lambda(V) < \infty$,

$$\lim_{t \to +\infty} \; \{\Psi_1(r_*(t)) \vee 1\} \; \int_X \lambda(dx) \, \|P^t(x, \cdot) - \pi(\cdot)\|_{\Psi_2(f_*) \vee 1} = 0, \tag{3.5}$$

where r_* and f_* are given by (3.2) and $\mathcal{I}$ is defined in Section 3.1. Furthermore, there exist finite constants $C_{\Psi,i}$ such that for all $t \geq 0$ and all $x \in X$,

$$\{\Psi_1(r_*(t)) \vee 1\} \quad \|P^t(x, \cdot) - \pi(\cdot)\|_{\Psi_2(f_*) \vee 1} \leq C_{\Psi,1} \, V(x), \tag{3.6}$$

$$\int_0^\infty \{\Psi_1(r_*(t)) \vee 1\} \quad \|P^t(x, \cdot) - P^t(y, \cdot)\|_{\Psi_2(f_*) \vee 1} \, dt \leq C_{\Psi,2} \, \{V(x) + V(y)\}; \tag{3.7}$$

and if $\partial[\Psi_1(r_*)] \in \Lambda$, there exists a finite constant $C_{\Psi,3}$ such that for all $t \geq 0$,

$$\int_0^\infty \{\partial[\Psi_1(r_*)](t) \vee 1\} \quad \|P^t(x, \cdot) - \pi(\cdot)\|_{\Psi_2(f_*) \vee 1} \, dt \leq C_{\Psi,3} \, V(x). \tag{3.8}$$

The limit (3.5) is a direct application of [11, Theorem 1] while (3.6) to (3.8) are, to our best knowledge, new results. The proof of this theorem is detailed in Appendix A.

As already commented in [11], Eq. (3.5) shows that the rate of convergence and the norm in which convergence occurs have to be balanced : if Ψ_1 strongly increases at infinity then Ψ_2 slowly increases (see [19] and the comments in Section 3.1). Hence, the stronger the norm, the weaker the rate and conversely. The maximal rate of convergence is achieved with the total variation norm ($\Psi_2 \circ f_* = \mathbf{1}$) and the minimal one ($\Psi_2 \circ r_* = \mathbf{1}$) is achieved with the f_*-norm. Hence, the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ directly provides two major informations: the largest rate of convergence $r_* = \phi \circ H_\phi^{-1}$ is given by the concave function ϕ and the largest norm of convergence $\|\cdot\|_{f_*}$ is given by the pair (ϕ, V).

Eqs. (3.6) to (3.8) are, to our best knowledge, the first results that address the dependence upon the initial point in the ergodic behavior. When applied to discrete-time Markov chains, (3.6) to (3.8) coincide with resp. [31, Theorems 2.1, 4.1, 4.2] (the dependence upon x can be read from the proof of these theorems; the details are also provided in [9, Chapter 3]). These results for the discrete-time case and the definition of the set $\mathcal{S}_\psi$ in [11, Theorem 1] suggest that in (3.6), the minimal dependence in the starting value x is of the form $G_C(x, \Psi_2(f_*), \Psi_1(r_*); \delta)$. Similar expressions can be predicted for (3.7) and (3.8). The proof of this assertion and the explicit construction of the constants $C_{\Psi,i}$ in terms of the quantities appearing in the assumptions are beyond the scope of this paper. Currently in progress is work on explicit control of subgeometric ergodicity for strong Markov processes.

In the examples given in Section 4, we will see that the pair (ϕ, V) that solves $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ is not unique. Roughly speaking, we read from Theorem 3.10 that ϕ is related to the rate of convergence in total variation norm, while V is the dependence upon the initial point in the control of convergence. As a consequence, the rate of convergence $r_\phi(t)$ and the dependence $V(x)$ can be balanced to make the bounds (3.6) to (3.8) minimal. In Section 4, we will give some examples (on $X = \mathbb{R}^n$), where both a pair of polynomially increasing functions and a pair of subgeometrically increasing functions can be found. One then immediately remarks that the

stronger the control in the initial point is, the stronger the decay in time is for a given norm. It stresses once again the interest for exact constant in our controls to decide which "ergodicity" to use to reach a certain level. The fact that the pair (ϕ, V) is not unique shows that the drift condition only provides an upper bound of the true rate of convergence. Nevertheless, in many applications, we are able to prove that the true rate belongs to the exhibited class of rate functions (see for example, section 4.2).

3.4 Generator and drift inequality (3.1)

The drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ may not be easy to derive since it is couched in terms of the process itself. The main goal of this section is to provide an easier path to ensure subgeometric ergodicity, which is moreover the usual form of conditions adopted on earlier paper to address different classes of stability. Namely we will use the formalism of the extended generator [3, Def. 1.15.15].

Let $\mathcal{D}(\mathcal{A})$ denote the set of measurable functions $f : \mathsf{X} \to \mathbb{R}$ with the following property: there exists a measurable function $h : \mathsf{X} \to \mathbb{R}$ such that the function $t \mapsto h(X_t)$ is integrable $\mathbb{P}_x$-a.s. for each $x \in \mathsf{X}$ and the process

$$t \mapsto f(X_t) - f(X_0) - \int_0^t h(X_s)ds \qquad (3.9)$$

is a $\mathbb{P}_x$-local martingale for all x. Then we write $h = \mathcal{A}f$, and f is said in the domain of the extended generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ of the process X. The condition (3.1) looks like a Dynkin formula. This is the reason why we want it to hold as widely as possible, thus justifying the interest in the extended generator concept.

Theorem 3.11. *Assume that there exist a closed petite set C, a cad-lag function $V : \mathsf{X} \to [1, \infty)$ with $V \in \mathcal{D}(\mathcal{A})$, an increasing differentiable concave positive function $\phi : [1, \infty) \to (0, \infty)$ and a constant $b < \infty$ such that for all $x \in \mathsf{X}$,*

$$\mathcal{A}V(x) \leq -\phi \circ V(x) + b\mathbf{1}_C(x). \qquad (3.10)$$

Then $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ holds.

The proof is in Section A.7. The extended generator is less restrictive than the infinitesimal generator $\tilde{\mathcal{A}}$: if f is in the domain of $\tilde{\mathcal{A}}$, then the process (3.9) is a martingale and f is in the domain of $\mathcal{A}$ (see e.g. [3, Proposition 1.14.13]). In particular, it is often quite difficult to characterize the domain of $\tilde{\mathcal{A}}$ but there may be (and are, in the applications of Section 4) easily checked sufficient conditions for membership of $\mathcal{D}(\mathcal{A})$.
This drift condition naturally inserts in the existing literature, that addresses criteria for non-explosivity, recurrence, polynomial ergodicity, geometric and uniform ergodicity. More precisely, Meyn and Tweedie provide conditions for non-explosion, recurrence, positive-Harris recurrence

and V-ergodicity at a subgeometric rate, respectively of the form

$$\mathcal{A}V(x) \leq cV(x), \tag{3.11}$$

$$\mathcal{A}V(x) \leq c\mathbf{1}_C(x), \tag{3.12}$$

$$\mathcal{A}V(x) \leq -cf(x) + b\mathbf{1}_C(x), \tag{3.13}$$

$$\mathcal{A}V(x) \leq -cV(x) + b\mathbf{1}_C(x) \tag{3.14}$$

for some positive constants $b, c < \infty$ and a measurable function $f \geq 1$ (see [24, Conditions (CD0) to (CD3)]; see also [6] for the condition (3.14)). These criteria are similar to some conditions provided by [16] for the stability of stochastic differential equations. The drift inequality (3.14) is the limit of our approach, since it corresponds to (3.10) with $\phi(v) \propto v$.

In a recent work, Fort and Roberts [11] considered a family of drift condition that implies f-ergodicity at a polynomial rate : namely, there exist $0 < \alpha < 1$, $b > 0$ such that for all $\alpha \leq \eta \leq 1$, there exists $c_\eta > 0$ such that

$$\mathcal{A}V^\eta(x) \leq -c_\eta V^{\eta-\alpha}(x) + b1_C(x). \tag{3.15}$$

Our drift condition (3.10) with $\phi(v) \propto v^{1-\alpha}$ yields the same results as those provided in [11, Theorem 1, Lemma 25, Proposition 26] (see Theorem 3.10 and Proposition 3.1). Hence, the drift inequality (3.10) that addresses subgeometric ergodicity generalizes the criterion for polynomial ergodicity proposed by Fort-Roberts. The comparison of the Fort-Roberts nested drift conditions (3.15) and our single drift condition can be more explicit when $V \in \mathcal{D}(\mathcal{A})$ and the process (3.9) is a $\mathbb{P}_x$-martingale for all x. In that case, it is easily seen that the single drift condition implies the nested drift conditions. The martingale property is equivalent to

$$t \mapsto \exp\left(\ln V(X_t) - \ln V(X_0) - \int_0^t H(\ln V)(X_s)ds\right)$$

is a $\mathbb{P}_x$-martingale for all x, where $H(\ln V) = V^{-1}\mathcal{A}V$ [8]. Furthermore, $H(\ln V) \leq -g + s$ if and only if

$$t \mapsto \exp\left(\ln V(X_t) - \ln V(X_0) - \int_0^t \{-g(X_u) + s(X_u)\}du\right)$$

is a $\mathbb{P}_x$-supermartingale for all x [8]. As a consequence, if V^η is in the domain of $\mathcal{A}$ for all $0 \leq \eta \leq 1$ then the Jensen's inequality yields $H(\eta \ln V) \leq \eta \exp(-\alpha \ln V) + b\eta \exp(-\ln V)1_C$ which in turn implies (3.15).

3.5 Resolvent and drift inequality (3.1)

One of the approaches for studying the stability and ergodic theory of continuous time Markov processes consists in making use of the associated discrete time resolvent chains. This allows to take profit of the analysis of discrete time Markov chains which is quite well understood ([25, 22]) and then to transfer properties established in terms of the resolvent or "generalised resolvent" kernel (see for e.g. [21]) to the Markov process itself. Following the discussion (done

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for exponentially ergodic Markov process) by Down-Meyn-Tweedie [6, Th.5.1] and extending it to the subgeometric case, we will now link the drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b}, \beta)$ associated to the Markov process to a drift condition associated to the discrete time resolvent chain.

More precisely, define, for $\beta > 0$, the resolvent kernel R_β by $R_\beta(x, A) = \int_0^\infty \beta e^{-\beta t} P^t(x, A) dt$ and consider the following drift condition associated to the resolvent kernel.

$\check{\mathbf{D}}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b}, \beta)$: There exist a petite set C, a function $V : \mathsf{X} \to [1, \infty)$, an increasing differentiable concave positive function $\phi : [1, \infty) \to (0, \infty)$ and a constant $b < \infty$ such that for any $x \in \mathsf{X}$,
$$R_\beta V(x) \leq V(x) - \phi \circ V(x) + b\mathbf{1}_C(x). \tag{3.16}$$

The following result ensures that drift conditions expressed in terms of the resolvent kernel or of the Markov process are essentially equivalent. This theorem parallels Theorem 5.1. by Down-Meyn-Tweedie [6] for exponentially ergodic Markov processes.

Theorem 3.12. (i) Assume $\check{\mathbf{D}}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b}, \beta)$ where C is a closed set and $R_\beta V$ is a cad-lag function. Then $\mathbf{D}(\mathbf{C}, \mathbf{R}_\beta \mathbf{V}, \beta\phi, \beta\mathbf{b})$ holds.

(ii) Assume $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ with $\sup_C V < \infty$. Then, for all $\epsilon > 0$, there exists a constant c such that for all $x \in \mathsf{X}$,
$$W(x) \leq (1 + \epsilon)V(x) + c \quad and \quad \lim_{t \to \infty} \frac{r_{\check{\phi}}(t)}{r_\phi((1 + \epsilon)t)} = 1 + \epsilon$$

such that $\check{\mathbf{D}}(\check{\mathbf{C}}, \mathbf{W}, \check{\phi}, \check{\mathbf{b}}, \beta)$ holds.

The proof is given in Section A.8.

4 Examples

In this section, $\mathsf{X} = \mathbb{R}^n$. Vectors are intended as column vectors, $|x|$ and $\langle \cdot, \cdot \rangle$ denote respectively the Euclidean norm and the scalar product. For a matrix a, $|a| = \left(\sum_{i,j} a_{i,j}^2 \right)^{1/2}$, $\mathrm{Tr}(a)$ stands for the trace of the matrix and a' the matrix transpose. Id_n is the $n \times n$ identity matrix. If V is a twice continuously differentiable function with respect to $x \in \mathbb{R}^n$, ∂V (or $\partial_x V$ when confusion is possible) denotes its gradient, and $\partial^2 V$ its Hessian.
For a set A, A^c is its complement in $\mathbb{R}^n$.

Four applications are considered: we first analyze general elliptic diffusions on $\mathbb{R}^n$ such that the drift coefficient verifies a contraction condition of the form $\langle b(x), x \rangle \leq -r|x|^{1-p}$ for all large x, where $0 < p < 1$. We then consider a Langevin diffusion on $\mathbb{R}^n$ having an invariant distribution which is super-exponential in the tails, and show that the rate of convergence can be modified by "heating" the diffusion. The method is however not limited to elliptic diffusions but can also

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be of use in the hypoelliptic case: we consider as an illustration a simple stochastic damping Hamiltonian system which cannot be exponentially ergodic but is shown to be subexponentially ergodic. We finally study a compound Poisson-process driven Ornstein-Uhlenbeck process when the distribution of the jump is heavy tailed.

Queuing theory is another important field of application for our theory. We do not discuss here this field of applications. This will be done in a forthcoming paper, which will also include a comparison of our results to those by [2, 34]. Techniques in Dai-Meyn [2] differ from ours since they are based on fluid limits. Concerning [34], our conditions are more general; indeed the authors assume that there exists a state x_0 such that whenever the Markov process hits x_0, it will sojourn there for a random time that is positive with probability 1, [34, Assumption 1.1]. This assumption makes their results unavailable for the applications we now consider.

4.1 Elliptic diffusions on $\mathbb{R}^n$

Consider the stochastic integral equation of the form

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s, \tag{4.1}$$

where $X_t \in \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are measurable functions, and $\{B_t\}_t$ is a n-dimensional Brownian motion. Assume that $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ are functions satisfying

A1 σ is bounded and b and σ are locally Lipschitz : for any $l > 0$, there exists a finite constant c_l such that for all $|x| \leq l, |y| \leq l$,

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq c_l|x - y|. \tag{4.2}$$

Let $a(x) = \sigma(x)\sigma(x)'$ be the diffusion matrix. We assume that

A2 (i) $a(x)$ is non-singular : the smallest eigenvalue of the diffusion matrix $a(x)$ is bounded away from zero in every bounded domain.

(ii) there exist $0 < p < 1$, $r > 0$ and M such that for all $|x| \geq M$, $\langle b(x), x \rangle \leq -r|x|^{1-p}$.

Note that under A1, $\Lambda = n^{-1} \sup_{x \in \mathbb{R}^n} \text{Tr}(a(x))$ and $\lambda_+ = \sup_{x \neq 0} \langle a(x)\frac{x}{|x|}, \frac{x}{|x|} \rangle$ are finite. Moreover, since under A1 σ is continuous, the assumption A2(i) is equivalent to the condition $\det(\sigma(x)) \neq 0$ for all x.

Under A1, it is possible to define continuous functions b_l and σ_l that satisfy the at most linear increasing
$$|b_l(x)| + |\sigma_l(x)| \leq c_l(1 + |x|), \qquad \forall x \in \mathbb{R}^n,$$
the Lipschitz condition (4.2) on the whole state space, and are such that $b_l = b$ and $\sigma_l = \sigma$ on the cylinder $\{x \in \mathbb{R}^n, |x| < l\}$. The stochastic equation (4.1) has a unique t-continuous solution

$\{X_t^{(l)}\}_t$, when b and σ are replaced by b_l and σ_l [16, Theorem 3.3.2]. The first exit times of $\{X_t^{(m)}\}_t$ from $\{x \in \mathbb{R}^n, |x| < l\}$ are identical for all $m \geq l$ (and is thus denoted ζ_l). This allows the construction of a process $\{X_t\}_t$ that satisfies (4.1) up to the explosion time $\zeta = \lim_l \zeta_l$. If $\zeta = +\infty$ a.s., $\{X_t\}_t$ is a.s. defined for all $t \geq 0$ and the process is said regular. Under the stated assumptions, an easy to check sufficient condition for regularity relies on the operator L that acts on function $V : \mathbb{R}^n$, $x \mapsto V(x)$ that are twice continuously differentiable with respect to x:

$$LV(x) = \langle b(x), \partial V(x) \rangle + \frac{1}{2} \mathrm{Tr}\left(a(x) \ \partial^2 V(x)\right). \tag{4.3}$$

The process is regular if there exists a non-negative twice-continuously differentiable function V on $\mathbb{R}^n$ such that for some finite c, $LV \leq cV$ on $\mathbb{R}^n$ and $\inf_{|x|>R} V(x) \to \infty$ as $R \to \infty$ [16, Theorem 3.4.1.]. Under A2(ii), it is trivial to verify that by setting $V(x) = 1 + |x|^2$,

$$LV(x) \leq \begin{cases} -2r|x|^{1-p} + n\Lambda, & \text{if } |x| \geq M, \\ 2M \ \sup_{|x| \leq M} |b(x)| + n\Lambda & \text{otherwise.} \end{cases} \tag{4.4}$$

This shows that the process is regular. Consequently, there exists a solution to (4.1), which is an almost surely continuous stochastic process and is unique up to equivalence. This solution is an homogeneous Markov process whose transition functions are Feller functions [16, Theorem 3.4.1]. Hence, it is strongly Markovian, as a right-continuous Markov process with Feller transition functions. We now discuss the existence of an irreducible skeleton P^m and the petiteness property of the compact sets. All of these properties deduce from the existence of an unique invariant probability distribution π.

Proposition 4.1. *Under A1-A2, X possesses an unique invariant probability measure π. π is a maximal irreducibility measure and any skeleton P^m is irreducible. Furthermore, the compact sets are closed petite sets.*

Proof. By (4.4), [16, Theorem 3.7.1] and its corollary 2 [16, p. 99], there exists a bounded domain U with regular boundary and a finite constant c such that for all $x \in U^c$, $\mathbb{E}_x[T_{U^c}] < \infty$ and for any compact $\mathcal{K} \subset \mathbb{R}^n$, $\sup_{x \in \mathcal{K}} \mathbb{E}_x[T_{U^c}] < \infty$, where

$$T_U = \inf\{t \geq 0, X_t \notin U\}.$$

Since the diffusion matrix $a(x)$ is non-singular, we deduce from [16, Theorem 4.4.1 and Corollary 2 p.123] that the process possesses an unique invariant probability distribution π. [16, Lemma 4.6.5] implies that any skeleton is ϕ-irreducible, with an irreducibility measure absolutely continuous with respect to the Lebesgue measure. By [16, Lemma 4.6.1], the support of π has non-empty interior; since the process is ψ-irreducible and has the Feller property, all compact subsets of $\mathbb{R}^n$ are petite (this assertion can be proved in exactly the same way as in the discrete-parameter case [22, Proposition 6.2.8]). $\square$

Under A1-2, it si easily checked that any twice continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is in the domain of $\mathcal{A}$ and $LV(x) = \mathcal{A}V(x)$ for all $x \in \mathbb{R}^n$. Observe indeed that $t \mapsto LV(X_t)$

is integrable $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}^n$ and $t \mapsto V(X_t) - V(X_0) - \int_0^t LV(X_s)ds$ is a right-continuous local martingale. Hence $V \in \mathcal{D}(\mathcal{A})$ and $LV = \mathcal{A}V$. We now establish drift inequalities for different test functions V.

Proposition 4.2. *Assume A1-2. Let $V : \mathbb{R}^n \to [1, +\infty)$ be a twice continuously differentiable function such that $V(x) = \exp(\iota |x|^m)$ outside a compact set, for some $0 < m < 1$ and $\iota > 0$. Then $\sup_{|x| \le M} \mathcal{A}V(x) < \infty$ and for all $|x| \ge M$,*

(i) If $0 < m < 1 - p$,

$$\mathcal{A}V(x) \le -\iota^{\frac{1+p}{m}} mr \left[\ln V(x)\right]^{1 - \left(\frac{1+p}{m}\right)} V(x) \quad (1 + o(1));$$

(ii) If $m = 1 - p$,

$$\mathcal{A}V(x) \le -\iota^{\frac{1+p}{1-p}}(1 - p) \left\{r - (1/2)\lambda_+\iota(1 - p)\right\} \left[\ln V(x)\right]^{-2\frac{p}{1-p}} V(x) \quad (1 + o(1)).$$

Proof. Under the stated assumptions, $\sup_{\{x, |x| \le M\}} \mathcal{A}V(x) < \infty$. By definition of $\mathcal{A}$, we have for all $|x| \ge M$,

$$\mathcal{A}V(x) \le -\iota m \left(r - (1/2)\lambda_+\iota m |x|^{p+m-1}\right) |x|^{m-1-p} V(x) + (1/2)\iota m n \Lambda |x|^{m-2} V(x).$$

$\square$

As a direct application of Proposition 3.9 and Theorem 3.1(ii), we have

Theorem 4.3. *Assume A1-2.*

(i) For all $\iota > 0$ such that $r - (1/2)\lambda_+\iota(1 - p) > 0$,

$$\int \pi(dx) \exp(\iota|x|^{1-p}) < \infty,$$

where π is the invariant probability distribution of the Markov process that solves (4.1).

(ii) There exists a closed petite set C such that for any $0 < m < 1 - p$, $0 < \iota_1 < \iota_2$ and $\delta > 0$, there exists a finite constant c such that

$$\mathbb{E}_x \left[\exp(\iota_1 \{\tau_C(\delta)\}^{\frac{m}{1+p}})\right] \le c \exp(\iota_2 |x|^m). \tag{4.5}$$

If $m = 1 - p$, (4.5) still holds for any $0 < \iota_1 < \iota_2$ such that $r - (1/2)\iota_2\lambda_+(1 - p) > 0$.

The results of Theorem 4.3 can be compared to those by [20], where subexponential ergodicity in total variation norm of a diffusion satisfying the conditions A1-2 is addressed. The technique used in [20] is based on the coupling method. Theorem 4.3(i) states the same result as [20, Lemma 3]. Nevertheless, Theorem 4.3(ii) yields a stronger control of delayed return-time to

a closed petite set than those obtained in [20, Theorem 5]. They show that for all $0 < \alpha < (1/2)(1-p)$ there exists a constant c_α such that

$$\mathbb{E}_x \left[\exp(\tau_C(\delta)^\alpha) \right] \leq c_\alpha \exp(|x|^{2\alpha}),$$

and this remains valid for $\alpha = (1-p)/2$ if $r - (1/2)\lambda_+(1-p) > 0$. Theorem 4.3(ii) claims that for all $0 < \alpha < (1-p)(1+p)^{-1}$ and $\iota > 1$, $\mathbb{E}_x \left[\exp(\tau_C(\delta)^\alpha) \right] \leq c_\alpha \exp(\iota |x|^{(1+p)\alpha})$ and for $\alpha = (1-p)(1+p)^{-1}$, $\mathbb{E}_x \left[\exp(\iota_1\tau_C(\delta)^\alpha) \right] \leq c_\alpha \exp(\iota_2 |x|^{(1+p)\alpha})$ for all $0 < \iota_1 < \iota_2$ such that $r - (1/2) \iota_2\lambda_+(1-p) > 0$.

As a direct application of Theorem 3.10, we obtain the following results for f-ergodicity at a subgeometric rate.

Theorem 4.4. *Assume A1-2 and let π be the invariant probability distribution of the Markov process that solves (4.1). Then the process is subgeometrically f-ergodic: for any $x \in \mathbb{R}^n$, the limits (3.5) to (3.8) hold with $V(x) \sim \exp(\iota|x|^{1-p})$ for some positive ι such that $r - 0.5\lambda_+\iota(1-p) > 0$, $f_*(x) \sim |x|^{-2p} \exp(\iota|x|^{1-p})$ and $r_*(t) \sim t^{-2p/(1+p)} \exp(\{\iota't\}^{(1-p)/(1+p)})$ where*

$$\iota' = \iota^{\frac{1+p}{1-p}}(1+p)\left\{r - (1/2)\lambda_+\iota(1-p)\right\}.$$

In [20], only the convergence in total variation norm of the semi-group $\{P^t\}_{t\geq0}$ to the invariant probability π is addressed: is is established that the process is ergodic at the rate $r_*^{\mathrm{M}}(t) \propto \exp(\delta t^{(1-p)/2})$ for some $\delta > 0$, and in that case, the dependence upon the initial point in (3.5) is $V^{\mathrm{M}}(x) \sim \exp(\delta|x|^{1-p})$. Theorem 4.4 improves these results and also provides rates of convergence in f-norm for unbounded functions f.

We reported in Theorem 4.4 the values (V, f_*, r_*) that yield the best rate of convergence in total variation norm. Proposition 4.2 shows that one could establish the drift inequality (3.10) with $V(x) \sim \exp(\iota|x|^m)$ for some $0 < m < 1 - p$; this would imply the limits (3.5) to (3.8) with $V(x) \sim \exp(\iota|x|^m)$, $f_*(x) \sim |x|^{m-1-p} \exp(\iota|x|^m)$ and $r_*(t) \sim t^{(m-1-p)/(1+p)} \exp(\iota'|x|^{m/(1+p)})$ for all $0 < \iota' < \iota$. We thus obtain a weaker maximal rate function r_*, and a weaker maximal norm $\| \cdot \|_{f_*}$, but this has to be balanced with the fact that the dependence upon the initial value (*i.e.* the quantity $V(x)$) is weaker too. Similarly, polynomially increasing controls $V(x)$ could be considered, thus limiting the rate r_* (resp. the function f_*) to the class of the polynomially increasing rate functions (resp. to the class of the polynomially increasing function). These discussions illustrate the fact that the pair (ϕ, V) that solves (3.10) is not unique, and this results in balancing the pair (r_*, f_*) and the dependence upon the initial value x.

4.2 Langevin tempered diffusions on $\mathbb{R}^n$

Let $\pi : \mathbb{R}^n \to (0, \infty)$ satisfying

B1 π is, up to a normalizing constant, a positive and thrice continuously differentiable density on $\mathbb{R}^n$, with respect to the Lebesgue measure.

Let $\sigma(x) = |\ln \pi(x)|^d$ for some $d > 0$ and define the diffusion matrix by $a(x) = \sigma^2(x)\mathrm{Id}_n$, and the drift vector by $b(x) = (b_1(x), \cdots, b_n(x))'$ where

$$b_i(x) = (1/2) \sum_{j=1}^{n} a_{ij}(x)\, \partial_{x_j} \log \pi(x) + (1/2) \sum_{j=1}^{n} \partial_{x_j} a_{ij}(x), \qquad 1 \le i \le n.$$

Observe that since π is defined up to a normalizing constant, we can assume that $\sigma(x) > 0$ for all x. Our objective is to study the ergodicity of the solution to the stochastic integral equation

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \tag{4.6}$$

where $\{B_t\}_t$ is a n-dimensional Brownian motion. This diffusion is the so-called Langevin diffusion and the drift vector b is defined in such a way that π is, up to a multiplicative constant, the density of the unique invariant probability distribution. Note that this model is not a particular case of the elliptic diffusion of section 4.1 since here, σ may be an unbounded function ($\sigma = |\ln \pi(x)|^d$).

Fort and Roberts investigate the behavior of these diffusions when π is polynomially decreasing in the tails and address ergodicity in total variation norm and in f-norm as well [11]. They consider the case $\sigma(x) = \pi^{-d}(x)$ ($d > 0$) and show that the rate of convergence in total variation norm and in f-norm for $f(x) \sim \pi^{-\kappa}(x)$ ($\kappa > 0$) depends on d. When d is lower than some critical temperature d_*, the process is ergodic at a polynomial rate, and when d is larger than d_*, the process is uniformly ergodic in total variation norm and geometrically ergodic otherwise [11, Theorem 16]. Fort and Roberts thus proved that the rate of convergence can be improved by choosing a diffusion coefficient σ which is small when the process is close to the modes of π and big when it is far from the modes. The objective of this section is to investigate the case when π is super-exponentially decreasing in the tails. We assume that

B2 there exists $0 < \beta < 1$ such that for all large $|x|$,

$$|x|^{1-\beta} \left\langle \partial \ln \pi(x), \frac{x}{|x|} \right\rangle < 0,$$

$$0 < \liminf_{x \to \infty} |\partial \ln \pi(x)|| \ln \pi(x)|^{1/\beta - 1} \le \limsup_{x \to \infty} |\partial \ln \pi(x)|| \ln \pi(x)|^{1/\beta - 1} < \infty,$$

$$\limsup_{x \to \infty} \mathrm{Tr}\left(\partial^2 \ln \pi(x)\right) |\partial \ln \pi(x)|^{-2} = 0.$$

The class of density π described by B1-2 contains densities that are super-exponential in the tails. The Weibull distribution on $(0, \infty)$ with density $\pi(x) \propto x^{\beta - 1} \exp(-\alpha x^\beta)$ satisfies B2. For multidimensional examples, see e.g. [18, 28, 10]. Following the same steps as in Section 4.1, we can prove that under B1-2 and provided the process is regular, there exists a solution to (4.6) which is an almost surely continuous stochastic process and is unique up to equivalence. This solution is an homogeneous strong Markov process whose transition functions are Feller functions. Under B2, the process is regular whatever $d > 0$; this can be proved as in the previous section (by choosing $V = 1 + \pi^{-2}$, see (4.7) below).

These assumptions also imply that π is (up to a scaling factor) the density of an invariant distribution of the diffusion process, any skeleton chain is ψ-irreducible and compact sets are closed petite sets ([11, Proposition 15]).

Let $V : \mathbb{R}^n \to [1, \infty)$ be a twice-continuously differentiable function such that $V(x) = 1 + \pi^{-\kappa}(x)$ outside a compact set; then $\mathcal{A}V(x) = LV(x) = \ell_1(x) + \ell_2(x)$ where L is the diffusion operator (4.3) and for large $|x|$,

$$\ell_1(x) = -\frac{\kappa(1-\kappa)}{2} \frac{\pi^{-\kappa}(x)}{1 + \pi^{-\kappa}(x)} \left(\frac{|\partial \ln \pi(x)|}{|\ln \pi(x)|^{1-1/\beta}} \right)^2 |\ln \pi(x)|^{2(d+1-1/\beta)} V(x), \qquad (4.7)$$

and $\ell_2(x) = o(\ell_1(x))$. In [30, Theorem 3.1], it is established that the process is geometrically ergodic if and only if $d \geq 1/\beta - 1$. From (4.7), we are able to retrieve these results and we also prove that when $0 \leq d < 1/\beta - 1$, the process is subgeometrically ergodic. Observe indeed that for large $|x|$, (4.7) and B2 imply

$$\mathcal{A}V(x) \leq -c_\kappa [\ln V(x)]^{-\alpha} V(x), \quad \text{where } \alpha = 2(1/\beta - 1 - d), \quad \text{and } c_\kappa > 0 \Longleftrightarrow 0 < \kappa < 1.$$

Hence, if $\alpha \leq 0$, the process is V-geometrically ergodic [21, Theorem 6.1] (see also section 3.4); if $\alpha > 0$, it is subgeometrically ergodic as a consequence of Theorems 3.10 and 3.11.

A polynomially increasing drift function can also be considered: we can assume without loss of generality that for large x, $\ln \pi(x) < 0$ since π is defined up to a multiplicative constant. We thus set $V(x) = 2 + \text{sign}(\kappa) (-\ln \pi(x))^\kappa$ outside a compact set. Then for large x,

$$\mathcal{A}V(x) \leq -\frac{|\kappa|}{2} (-\ln \pi(x))^{\kappa+1+2(d-1/\beta)} \left(\frac{|\partial \ln \pi(x)|}{|\ln \pi(x)|^{1-1/\beta}} \right)^2 (1 + o(1)),$$

and there exists a constant $c > 0$ such that for large x,

$$\mathcal{A}V(x) \leq -cV^{1-\alpha}(x), \quad \text{where } \alpha = 2\kappa^{-1}(1/\beta - d - (1/2)). \qquad (4.8)$$

First consider the case when $\kappa > 0$. If $1/\beta - 1 < d < 1/\beta - (1/2)$, the drift condition (4.8) and Theorems 3.10 and 3.11 yield polynomial ergodicity. For example, this implies convergence in total variation norm at the rate $r(t) \sim t^{1/\alpha-1}$. If $d = 1/\beta - (1/2)$, then $\alpha = 0$ and the process is geometrically ergodic. In the case when κ can be set negative and $1 - \alpha > 0$ *i.e.* when $d > 1/\beta - (1/2)$, the process is uniformly ergodic: there exist $\lambda < 1$ and a constant $c < \infty$ such that for all x,

$$\lambda^{-t} \|P^t(x, \cdot) - \pi(\cdot)\|_{\text{TV}} \leq c,$$

and the convergence does not depend on the starting point.

The above discussions are summarized in the following theorem. The first part (resp. third part) results from [30, Theorem 3.1] (resp. [21, Theorem 6.1]). The second assertion is a consequence of Theorem 3.10. The last assertion was already proved by [30, Theorem 3.1] for one-dimensional diffusions ($n = 1$).

Theorem 4.5. *Consider the Langevin diffusion on $\mathbb{R}^n$ solution to the equation (4.6) where the target distribution π satisfies B1-2.*

(i) If $0 \leq d < 1/\beta - 1$, the process fails to be geometrically ergodic.

(ii) If $0 \leq d < 1/\beta - 1$, the process is subgeometrically ergodic: the limits (3.5) to (3.8) hold with $V(x) \sim \pi^{-\kappa}(x)$, $f_*(t) \sim \pi^{-\kappa}(x) |\ln \pi(x)|^{-2(1/\beta - 1 - d)}$ and $\ln r_*(t) \sim c_\kappa t^{\beta/(2-\beta-2d\beta)}$ for all $0 < \kappa < 1$.

(iii) If $d \geq 1/\beta - 1$, then for all $0 < \kappa < 1$, the diffusion is V-geometrically ergodic with $V(x) = 1 + \pi^{-\kappa}(x)$.

(iv) If $d > 1/\beta - (1/2)$, the diffusion is uniformly ergodic.

This theorem extends earlier results to the multi-dimensional case and provides subgeometrical rates of convergence of the 'cold' Langevin diffusion, for a wide family of norms. We established that for a given $\pi^{-\kappa}$-norm, the minimal rate of convergence is achieved with $d = 0$ and in that case, the rate coincides with the rate of convergence of the symmetric random-walk Hastings-Metropolis algorithm ([4, Theorem 3.1]). This rate can be improved by choosing a diffusion matrix which is heavy where π is light and conversely. When d is larger than the critical value $d_* = 1/\beta - 1$, the process is geometrically ergodic; when d is lower that d_*, the process can not be geometrically ergodic and we prove that it is subgeometrically ergodic. The conclusions of Theorem 4.5 are similar to those of [11, Theorem 16], that address the case when π is polynomial in the tails.

We assumed that $\sigma = |\ln \pi|^d$. A first extension is to consider a sufficiently smooth function σ such that $\sigma(x) \sim |\ln \pi(x)|^d$ for large $|x|$; this yields similar conclusions and details are omitted. A second extension consists in the case when $\sigma(x) \sim \pi^{-d}(x)$. In this latter case, following the same lines, it is easily verified that for small enough values of d, the process is regular (the set of the admissible values is in the range $(0, 1/2]$), and the process is V-geometrically ergodic with a test function $V(x) \sim \pi^{-\kappa}(x)$, $\kappa > 0$. Details are omitted and left to the interested reader.

4.3   Stochastic damping Hamiltonian system

Both examples of the previous sections assumed that the diffusion process is elliptic. However the drift condition (3.10) enables us to consider also hypoelliptic diffusion that we will illustrate on the example of a simple stochastic damping Hamiltonian system, i.e. let x_t (resp. y_t) be the position (resp. the velocity) at time t of a physical system moving in $\mathbb{R}^n$

$$
\begin{aligned}
dX_t &= Y_t dt \\
dY_t &= \Sigma(X_t, Y_t) dB_t - (c(X_t, Y_t)Y_t + \partial_x U(X_t)) dt
\end{aligned}
\tag{4.9}
$$

where $-\partial_x U$ is some friction force, $-c(x,y)y$ is the damping force and $\Sigma(x,y)dB$ is a random force where (B_t) is a standard Brownian motion in $\mathbb{R}^n$. This system has been studied from the large and moderate deviations point of view by Wu [33] where he also establishes the exponential ergodicity under various set of assumptions.

As our goal is not to consider the model in its full generality but to illustrate the subexponential behavior of hypoelliptic diffusion, via the simple use of drift condition (3.10), hereafter

we will consider the particular (but also current in practice) case where the damping and random forces are constant $c(x, y) = c\,\mathrm{Id}_n$ and $\Sigma(x, y) = \sigma\,\mathrm{Id}_n$, c and σ being positive constants (as, if one is identically equal to 0, there is none of the usual ergodic properties such as positive recurrence). We will assume moreover that the potential U is lower bounded and continuously differentiable over $\mathbb{R}^n$. In this case, the system is known to have an unique invariant measure given by

$$\pi(dx, dy) = e^{-\frac{2c}{\sigma}H(x,y)}dxdy$$

where H is the Hamiltonian given by $H(x, y) = \frac{1}{2}|y|^2 + U(x)$.

Let us first ensure the existence of solutions and aperiodicity for the process $Z_t = (X_t, Y_t)$ via the following proposition due to Wu [33, Lemma 1.1, Proposition 1.2]

Proposition 4.6. *For every initial state $z = (x, y) \in \mathbb{R}^{2n}$, the SDE (4.9) admits an unique weak solution $\mathbf{P}_z$ which is non explosive. Moreover denoting $(P^t(z, dz'))_t$ the associated semi group of transition, we have that for every $t > 0$ and every $z \in \mathbb{R}^{2n}$, $P^t(z, dz') = p_t(z, z')dz'$ and $p_t(z, z') > 0$, $dz' - a.e.$ The density $p_t(z, \cdot)$ is moreover continuous, and the process is thus strongly Feller.*

As a consequence, the solution is a strong Markov process, all the skeletons are irreducible and compact sets are petite sets.

Let us build an example of polynomially ergodic stochastic damping Hamiltonian system in dimension 1. We rewrite the system as

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= \sigma dB_t - (cY_t + U'(X_t))dt, \end{aligned} \qquad (4.10)$$

and assume that U is C^2, and there exist $0 < p < 1$ and positive constants a, b such that for $|x|$ large enough

$$a|x|^{p-1} \leq U'(x) \leq b|x|^{p-1}. \qquad (4.11)$$

The fact that p is less than 1 implies that $(Z_t)_{t \geq 0}$ cannot be exponentially ergodic [33, Theorem 5.1]. We now exhibit a drift function satisfying (3.10). Consider positive constants α, β and a smooth positive function G such that for m, $1 - p < m \leq 1$, $G'(x) = |x|^m$ for large $|x|$; define a twice continuously differentiable function $V \geq 1$ such that for large x, y,

$$V(x, y) = \alpha(y^2/2 + U(x)) + \beta(G'(x)y + cG(x)).$$

By definition of $\mathcal{A}$, it holds

$$\mathcal{A}V_m(x, y) = \frac{1}{2}\sigma^2\,\partial_y^2 V_m(x, y) + y\,\partial_x V_m(x, y) - (cy + U'(x))\,\partial_y V_m(x, y)$$

so that

$$\begin{aligned} \mathcal{A}V_m(x, y) &= \frac{1}{2}\alpha\sigma^2 + y(\alpha U'(x) + \beta G''(x)y + \beta cG'(x)) - (cy + U'(x))(\alpha y + \beta G'(x)) \\ &= \frac{1}{2}\alpha\sigma^2 + (\beta G''(x) - c\alpha)y^2 - \beta G'(x)U'(x). \end{aligned}$$

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Fix $\delta < 0$; since $m \leq 1$, we choose β small enough so that $\beta G''(x) - c\alpha < \delta < 0$ for all large x. Furthermore, for all large $|x|$, $G'(x)U'(x) \geq b|x|^{p-1+m}$. Hence, there exist positive constants K, L such that

$$\mathcal{A}V_m(x,y) \leq K - L\, V_m(x,y)^{\frac{p-1+m}{m+1}}.$$

Condition (3.10) holds with $\phi_m(v) \propto v^{\frac{p-1+m}{m+1}}$ and $\frac{p-1+m}{m+1} < 1$. Application of the results of Section 3.3 now implies that the process $(Z_t)_{t \geq 0}$ is polynomially-ergodic.

Let $k \geq 1$ and define a twice continuously differentiable function $V_{m,k} \geq 1$ such that for large x, y

$$V_{m,k}(x,y) = V_m^k(x,y).$$

Then for large x, y, the above calculations yield

$$
\begin{aligned}
\mathcal{A}V_{m,k}(x,y) &= (\mathcal{A}V_m(x,y))V_m^{k-1}(x,y) + \frac{1}{2}\sigma^2 (\partial_y V_m(x,y))^2 V_m^{k-2}(x,y) \\
&= \left(\mathcal{A}V_m(x,y)) + \frac{1}{2}\sigma^2 \frac{(\partial_y V_m(x,y))^2}{V_m(x,y)} \right) V_m^{k-1}(x,y) \\
&\leq (K' - LV_m(x,y)^{\frac{p-1+m}{m+1}})V_m^{k-1}(x,y) \\
&\leq K'' - L'V_m^{\frac{p-1+m}{m+1}+k-1}
\end{aligned}
$$

for some positive constant K', K'', L'. This inequality is once again the condition (3.10) with $\phi_{m,k}(v) = v^{(\frac{p-2}{m+1}+k)k^{-1}}$. These discussions are summarized in the following Theorem.

Theorem 4.7. *Let U be a twice continuously differentiable function, lower bounded on $\mathbb{R}$ satisfying (4.11) for some $0 < p < 1$. Then $(Z_t)_{t \geq 0}$ is not exponentially ergodic but is polynomially ergodic : for any m such that $1 - p < m \leq 1$ and any $k \geq 1$, the limits (3.5-3.8) hold with V_{mk} defined above, $\phi_{m,k}(v) \propto v^{(\frac{p-2}{m+1}+k)k^{-1}}$, $f_* = \phi_{m,k} \circ V_{m,k}$ and $r_*(t) \propto t^{\frac{k(m+1)}{2-p}-1}$.*

Observe that the process $(Z_t)_{t \geq 0}$ is polynomially ergodic at any order and we strongly believe it is subexponentially ergodic. This sub exponential case is left to the interested reader. The multidimensional case is more intricate in the choice of the drift function and we do not pursue here in this direction.

This example shows that our conditions are sufficiently flexible to consider the hypoelliptic diffusions as well as the elliptic ones.

4.4 Compound Poisson-process driven Ornstein-Uhlenbeck process

In this section we consider an example of Fort-Roberts [11] where subgeometric ergodicity can be achieved where they only obtain polynomial ergodicity. Let us first recall the model. Let X be an Ornstein-Uhlenbeck process driven by a finite rate subordinator:

$$dX_t = -\mu X_t + dZ_t$$

and $Z_t = \sum_{i=1}^{N_t} U_i$, where $(U_i)_{i\geq 1}$ is a sequence of i.i.d.r.v. with probability measure F, and (N_t) is an independent Poisson process of rate λ. We suppose the recall coefficient μ to be positive. Remarking that only when F is sufficiently (even extremely) heavy tailed, X fails to be exponentially ergodic, Fort-Roberts [11] give conditions for which X is polynomially ergodic. Namely, denote G the law of the log jump sizes $(G(A) = F(e^A))$, and assume that for all $\kappa > 0$, $\int e^{\kappa x} dG(x) = +\infty$. Lemma 17 of Fort-Roberts then prove that X is not exponentially ergodic and give examples where X is positive recurrent and polynomially ergodic, namely when for some $r > 1$, $\int_0^\infty [\log(1+u)]^r F(du)$ is finite. Such assertion may be useful considering

$$F(dx) = \frac{C_k^{-1}}{x(\log(x))^k} dx \qquad k > 1$$

$$F(dx) = \frac{C_{\beta,c}^{-1} e^{-c(\log(x))^\beta}}{x} dx \qquad \beta \leq 1.$$

We shall strengthen their result by

Proposition 4.8. *Suppose that (X_t) is aperiodic and that for some $\delta < 1$, $\alpha > 0$*

$$\int_0^\infty e^{\alpha(\log(1+x))^\delta} F(dx) < \infty.$$

Then, the conclusions of Theorem 3.10 hold with $V(x) = e^{\alpha'(\log x)^{\delta'}}$ (and $\alpha' < \alpha$ if $\delta' = \delta$), and $\phi(v) = v^{(1-\delta')/\delta'}$, $r_*(t) = \mathfrak{a} t^{-(1+\delta')} e^{\mathfrak{b} t^{\delta'}}$, $f_* = \phi \circ V$.

Proof. We shall use the drift conditions introduced previously for the generator defined by for all functions V in the extended domain of the generator

$$\mathcal{A}V(x) = \lambda \int_0^\infty (V(x+u) - V(x)) F(du) - \mu x V'(x).$$

Choosing $V(x) = (\log(x))^r$, as in Fort-Roberts [11, Lemma 18], for sufficiently large x ensures the polynomial ergodicity at the previous rate. Consider now $V(x) = e^{\alpha'(\log x)^{\delta'}}$, so that

$$
\begin{aligned}
\mathcal{A}V(x) &= \lambda \int_0^\infty (e^{\alpha'(\log(x+u))^{\delta'}} - e^{\alpha'(\log x)^{\delta'}}) F(du) - \alpha' \delta' \mu \frac{e^{\alpha'(\log x)^{\delta'}}}{(\log x)^{1-\delta'}} \\
&\leq -\alpha'^{2-\delta'} \mu \delta' \frac{V}{(\log V)^{(1-\delta')/\delta'}} + b
\end{aligned}
$$

recalling that for large x

$$e^{\alpha'(\log(x+u))^{\delta'}} - e^{\alpha'(\log x)^{\delta'}} \sim \delta' \frac{e^{\alpha'(\log x)^{\delta'}}}{(\log x)^{1-\delta'}} \log(1+u/x);$$

the dominated convergence theorem ends the argument.

5 Skeleton chain and moderate deviations

We consider here an important field of application for this subgeometric rate, namely moderate deviations for bounded additive functionals of Markov process. In fact, Proposition 3.8 gives us more than a way to deal with subexponential ergodicity, it also implies a drift condition in the sense of Douc-Fort-Moulines-Soulier [4] which will enables us, at least in a bounded test function framework, to extend to the continuous time case some limit theorems tailored for the subexponential regime by Douc-Guillin-Moulines [5] such as moderate deviations. Moderate deviations are concerned with the asymptotic for centered g with respect to π and for $0 \leq t \leq T$ of

$$S_t^\epsilon = \frac{1}{\sqrt{\epsilon}h(\epsilon)} \int_0^t g(X_{s/\epsilon})ds$$

where as ϵ tends to 0, $h(\epsilon) \to \infty$ but $\sqrt{\epsilon}h(\epsilon) \to 0$, namely a regime between the large deviations and the central limit theorem. We may then state (proofs will be done in appendix.)

Theorem 5.1. *Assume that* $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ *holds with* $\sup_C V < \infty$, *and some skeleton chain is* ψ-*irreducible.*

(i) *For all* $m > 0$, *there exist a function* $W : \mathsf{X} \to [\phi(1), \infty)$, *a small set* $\tilde{C}$ *for the skeleton* P^m *and a positive constant* b' *such that* $\sup_{\tilde{C}} W$ *is finite, and on* X,

$$P^m W \leq W - \phi \circ W + b' \mathbf{1}_{\tilde{C}}, \qquad and \qquad \phi \circ V \leq W \leq \kappa V.$$

(ii) *Assume that* X_0 *is distributed as* μ *and* $\mu(V) < \infty$ *and that* g *is a bounded mapping from* X *to* $\mathbb{R}^n$ *with* $\pi(g) = 0$. *Suppose moreover that for all positive* a

$$\lim_{\epsilon \to 0} \frac{1}{h^2(\epsilon)} \log \left(\epsilon H_\phi^{-1} \left(\frac{a\, h(\epsilon)}{\sqrt{\epsilon}} \right) \right) = \infty$$

then $\mathbb{P}_\mu(S_\cdot^\epsilon \in \cdot)$ *satisfies a moderate deviation principle in* $C_0([0,1], \mathbb{R}^n)$ *(the space of continuous functions from* $[0,1]$ *to* $\mathbb{R}^n$ *starting from 0) equipped with the supremum norm topology, with speed* $\frac{1}{h^2(\epsilon)}$ *and rate function* I_g^h, *i.e. for all Borel set* $A \in C_0([0,1], \mathbb{R}^n)$

$$
\begin{aligned}
-\inf_{\gamma \in int(A)} I_g^h(\gamma) \quad &\leq \quad \liminf_{\epsilon \to 0} \frac{1}{h^2(\epsilon)} \log \mathbb{P}_\mu(S_\cdot^\epsilon \in A) \\
&\leq \quad \limsup_{\epsilon \to 0} \frac{1}{h^2(\epsilon)} \log \mathbb{P}_\mu(S_\cdot^\epsilon \in A) \leq -\inf_{\gamma \in cl(A)} I_g^h(\gamma)
\end{aligned}
$$

where I_g^h *is given by*

$$
I_g^h(\gamma) := \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 \sup_{\zeta \in \mathbb{R}^n} \left\{ \langle \dot{\gamma}(t), \zeta \rangle - \dfrac{1}{2}\sigma^2(\langle g, \zeta \rangle) \right\} dt & \text{if } d\gamma(t) = \dot{\gamma}(t)dt, \ \gamma(0)=0, \\[4mm] +\infty & \text{else,} \end{cases}
\tag{5.1}
$$

and

$$\begin{aligned}
\sigma^2(\langle g, \zeta \rangle) &= \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\pi \left(\int_0^n g(X_s) ds \right)^2 \\
&= 2 \int_{\mathsf{X}} \langle g, \zeta \rangle \int_0^\infty P_t \langle g, \zeta \rangle dt \ d\pi.
\end{aligned} \tag{5.2}$$

The proof is in Section A.9.

To the authors' knowledge, this moderate deviations result (even for bounded function) is the first one for Markov processes which are not exponentially ergodic. It extends then results of Guillin [15, Th 1.] or Wu [33, Th. 2.7] in the subexponential setting. As expected, all ranges of speed are not allowed for such a theorem but are limited by the ergodicity of the process (we refer to Douc-Guillin-Moulines [5, Sect. 4] for a complete discussion on this interplay). The extension of this moderate deviation principle to unbounded function is left for further research, as well as extension to inhomogeneous functional and averaging principle, those subjects needing particular tools and developments.

A Proofs

A.1 Proof of Theorem 3.1

Lemma A.1. *For any $M > 0$ and for any cad-lag function g,*

$$\lim_{\epsilon \to 0} \sum_{k=1}^{\lfloor M/\epsilon \rfloor} \left| \int_{\epsilon(k-1)}^{\epsilon k} (g(s) - g(t_{k-1})) ds \right| = 0. \tag{A.1}$$

Proof. First note that g is bounded since it is a cad-lag function. Let $\eta > 0$ be an arbitrary real. For any $x \in [0, M]$, there exists an interval $(x - \alpha, x + \alpha)$ such that

$$\forall s \in (x - \alpha, x), \ |g(s) - g(x-)| < \eta/2 \quad \text{and} \quad \forall s \in [x, x + \alpha), |g(s) - g(x)| < \eta/2$$

Thus, for any (u, v) in $(x - \alpha, x) \times (x - \alpha, x)$ or in $[x, x + \alpha) \times [x, x + \alpha)$, $|g(u) - g(v)| \leq \eta$. By compacity of $[0, M]$, there exists a finite number M_η of such intervals $(x_i - \alpha_i, x_i + \alpha_i)$ which covers $[0, M]$. Taking ϵ sufficiently small, it can be easily checked that any interval $[\epsilon(k-1), \epsilon k]$ is included in some interval $(x_i - \alpha, x_i + \alpha)$. Now, if some $x_i \in [\epsilon(k-1), \epsilon k]$, write $\sup_{u,v \in [\epsilon(k-1),\epsilon k]} |g(u) - g(v)| \leq 2 \sup_{x \in [0,M]} |g(x)|$. Otherwise, we have $\sup_{u,v \in [\epsilon(k-1),\epsilon k]} |g(u) - g(v)| < \eta$. Thus, since there is at most M_η intervals $[\epsilon(k-1), \epsilon k]$ which contain some x_i,

$$\sum_{k=1}^{\lfloor M/\epsilon \rfloor} \left| \int_{\epsilon(k-1)}^{\epsilon k} (g(s) - g(\epsilon(k-1))) ds \right| \leq 2 \sup_{x \in [0,M]} |g(x)| M_\eta \epsilon + \eta M$$

The proof follows by letting $\epsilon \to 0$ and by noting that η is arbitrary. $\qquad\square$

Proof. (Theorem 3.1) Proof of (i) is a direct application of the optional sampling theorem for a right continuous super-martingale (see e.g. [7, Theorem 2.13 p. 61])

$$s \mapsto V(X_s) - V(X_0) + \int_0^s \phi \circ V(X_u)du - b\int_0^s \mathbf{1}_C(X_u)du,$$

with the bounded $\mathcal{F}$-stopping time $\tau = \tau_C(\delta) \wedge M$ and by letting $M \to \infty$. We now prove (ii). Let $G(t,u) = H_\phi^{-1}(H_\phi(u)+t) - H_\phi^{-1}(t)$. Note that

$$\frac{\partial G(t,u)}{\partial u} = \frac{\phi \circ H_\phi^{-1}(H_\phi(u)+t)}{\phi(u)} = \frac{\phi \circ H_\phi^{-1}(H_\phi(u)+t)}{\phi \circ H_\phi^{-1}(H_\phi(u))} \tag{A.2}$$

$$\frac{\partial G(t,u)}{\partial t} = \phi \circ H_\phi^{-1}(H_\phi(u)+t) - \phi \circ H_\phi^{-1}(t) \tag{A.3}$$

By log-concavity of $\phi \circ H_\phi^{-1}$, for any fixed t, $u \mapsto \frac{\partial G(t,u)}{\partial u}$ is non increasing and thus, for any fixed t, the function $u \mapsto G(t,u)$ is concave.

Let $\epsilon > 0$. Write $t_k = \epsilon k$ and

$$N_\epsilon = \begin{cases} \sup\{k \geq 1; t_{k-1} < \tau_C(\delta)\} & \text{if } \tau_C(\delta) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Note that by (i), $\mathbb{P}_x(\tau_C(\delta) < \infty) = 1$. It is straightforward that $\tau_C(\delta) \leq \epsilon N_\epsilon$ and that ϵN_ϵ is a $\mathcal{F}$-stopping time. This implies that for any $M > \delta$,

$$
\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_C(\delta)\wedge M} \phi \circ H_\phi^{-1}(s)ds\right] - G(0,V(x)) &\leq \limsup_{\epsilon \to 0} \mathbb{E}_x\left[\int_0^{(\epsilon N_\epsilon)\wedge M} \phi \circ H_\phi^{-1}(s)ds\right] - G(0,V(x)) \\
&= \limsup_{\epsilon \to 0} \mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s)ds\right] - G(0,V(x)) \\
&\leq \limsup_{\epsilon \to 0} A(\epsilon) \tag{A.4}
\end{aligned}
$$

where

$$M_\epsilon := \lfloor M/\epsilon \rfloor,$$

$$A(\epsilon) := \mathbb{E}_x\left[G(\epsilon(N_\epsilon \wedge M_\epsilon), V(X_{\epsilon(N_\epsilon \wedge M_\epsilon)})) - G(0,V(x))\right] + \mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s)ds\right].$$

We now bound $\limsup_{\epsilon \to 0} A(\epsilon)$. First, write for any $\epsilon > 0$,

$$
\begin{aligned}
A(\epsilon) &= \mathbb{E}_x\left[\sum_{k=1}^{M_\epsilon} \{G(t_k, V(X_{t_k})) - G(t_{k-1}, V(X_{t_{k-1}}))\} \mathbf{1}_{\tau_C(\delta)>t_{k-1}}\right] + \mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s)ds\right] \\
&\leq \mathbb{E}_x\left[\sum_{k=1}^{M_\epsilon} \mathbb{E}\left[G(t_k, V(X_{t_k})) - G(t_{k-1}, V(X_{t_{k-1}}))\big| \mathcal{F}_{t_{k-1}}\right] \mathbf{1}_{\tau_C(\delta)>t_{k-1}}\right] + \mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s)ds\right] \tag{A.5}
\end{aligned}
$$

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where we have used that $\{\tau_C(\delta) > t_{k-1}\} \in \mathcal{F}_{t_{k-1}}$. Moreover, by concavity of $u \to G(t,u)$,

$$\mathbb{E}\left[G(t_k, V(X_{t_k})) - G(t_{k-1}, V(X_{t_{k-1}}))\big|\mathcal{F}_{t_{k-1}}\right]$$
$$\leq \frac{\partial G}{\partial u}(t_k, V(X_{t_{k-1}}))\mathbb{E}\left[V(X_{t_k}) - V(X_{t_{k-1}})\big|\mathcal{F}_{t_{k-1}}\right] + \int_{t_{k-1}}^{t_k} \frac{\partial G}{\partial t}(s, V(X_{t_{k-1}}))ds$$

Replacing by the expressions of the partial derivatives $\frac{\partial G}{\partial u}$ and $\frac{\partial G}{\partial t}$ given in (A.2) and (A.3) and inserting the resulting inequality in (A.5) yields, combining with $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$

$$A(\epsilon) \leq \mathbb{E}_x\left[\sum_{k=1}^{M_\epsilon} \phi \circ H_\phi^{-1}(H_\phi(V(X_{t_{k-1}})) + t_k)\left(-\frac{\int_{t_{k-1}}^{t_k} \phi \circ V(X_s)ds}{\phi(V(X_{t_{k-1}}))} + \epsilon\right)\mathbf{1}_{\tau_C(\delta)>t_{k-1}}\right]$$
$$+ \frac{b}{\phi(1)}\mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s+\epsilon)\mathbf{1}_C(X_s)ds\right]$$

Consider the first term of the rhs. By Fatou's lemma,

$$\limsup_{\epsilon \to 0} \mathbb{E}_x\left[\sum_{k=1}^{\lfloor M/\epsilon \rfloor} \phi \circ H_\phi^{-1}(H_\phi(V(X_{t_{k-1}})) + t_k)\left|-\frac{\int_{t_{k-1}}^{t_k} \phi \circ V(X_s)ds}{\phi(V(X_{t_{k-1}}))} + \epsilon\right|\right]$$
$$\leq \mathbb{E}_x\left[\phi \circ H_\phi^{-1}(H_\phi(\sup_{t \in [0,M]} V(X_t)) + M)\limsup_{\epsilon \to 0}\sum_{k=1}^{\lfloor M/\epsilon \rfloor}\left|\frac{\int_{t_{k-1}}^{t_k}\{\phi \circ V(X_s) - \phi \circ V(X_{t_{k-1}})\}ds}{\phi(1)}\right|\right] = 0 ,\ \blacksquare$$

by applying Lemma A.1 with $g(s) := \phi \circ V(X_s)$. Thus, using again Fatou's lemma,

$$\mathbb{E}_x\left[\int_0^{\tau_C(\delta) \wedge M} \phi \circ H_\phi^{-1}(s)ds\right] - G(0, V(x))$$
$$\leq \limsup_{\epsilon \to 0} A(\epsilon) \leq \frac{b}{\phi(1)}\limsup_{\epsilon \to 0}\mathbb{E}_x\left[\int_0^{\epsilon(N_\epsilon \wedge M_\epsilon)} \phi \circ H_\phi^{-1}(s+\epsilon)\mathbf{1}_C(X_s)ds\right]$$
$$\leq \frac{b}{\phi(1)}\mathbb{E}_x\left[\int_0^M \phi \circ H_\phi^{-1}(s)\mathbf{1}_C(X_s)\left(\limsup_{\epsilon \to 0}\mathbf{1}_{s \leq \epsilon N_\epsilon < \tau_C(\delta)+\epsilon}\right)ds\right]$$
$$= \frac{b}{\phi(1)}\mathbb{E}_x\left[\int_0^M \phi \circ H_\phi^{-1}(s)\mathbf{1}_C(X_s)\mathbf{1}_{s \leq \tau_C(\delta)}ds\right] = \frac{b}{\phi(1)}\int_0^\delta \phi \circ H_\phi^{-1}(s)ds$$

The proof follows by letting $M \to \infty$. $\square$

A.2 Proof of Proposition 3.4

The ψ-irreducibility results from [21, Theorem 1.1]. Under the stated assumptions, there exists a finite constant b' such that $RV(x) \leq V(x) + b'$ where R denotes the resolvent for the process $R(x,dy) = \int \exp(-t)P^t(x,dy)dt$. This shows that the set $\{V < \infty\}$ is absorbing for the R-chain,

and since R is ψ-irreducible, it is full or empty [22, Proposition 4.2.3]. Since $C \subset \{V < \infty\}$, this set is full.

Let B be a closed accessible petite set, the existence of which is proved in [21, Proposition 3.2(i)]. Since B is accessible, there exists t_0 and $\gamma > 0$ such that $\inf_{x \in C} \mathbb{P}_x (\tau_B \leq t_0) \geq \gamma$. Observe indeed that we can assume without loss of generality that C is ν_a-petite for some maximal irreducibility measure ν_a [21, Proposition 3.2]. Hence

$$0 < \nu_a(B) \leq \mathbb{P}_x (X_\xi \in B) \leq \mathbb{P}_x (X_\xi \in B, \xi \leq t_0) + \mathbb{P}_x (\xi > t_0) \leq \mathbb{P}_x (\tau_B \leq t_0) + \mathbb{P} (\xi > t_0),$$

where $\xi \sim a(dt)$ is independent of the process. Choose t_0 such that $\mathbb{P}(\xi > t_0) \leq 0.5\nu_a(B)$ and the existence of γ follows. In the proof of [21, Proposition 4.1], it is shown that for all $\delta > 0$, there exists a constant $c < \infty$ such that for all $x \in \mathsf{X}$,

$$\mathbb{E}_x [\tau_B] \leq \mathbb{E}_x [\tau_C(\delta)] + c.$$

Hence, by Theorem 3.1, there exists a constant $c < \infty$ such that $\mathbb{E}_x [\tau_B] \leq cV(x)$. This implies that the level sets $B_n = \{V \leq n\}$ are petite (see the proof of [21, Proposition 4.2]).

Since $\{V < \infty\}$ is full, $\cup_n B_n$ is full. This implies B_n is accessible for n large enough, and $C \subset B_{n_*}$ for some (and thus all) $n_* \geq \sup_C V$. Finally, since ν_a is a regular measure, there exists a compact set B such that $C \subseteq B \subseteq B_{n_*}$ and $\nu_a(B) > 0$. This concludes the proof.

A.3 Proof of Proposition 3.5

We can assume without loss of generality that $r \in \Lambda_0$ and we will do so.

By [11, Lemma 20], there exists a constant $\kappa < \infty$ such that

$$G_C(x, f, r; t) \leq \kappa^{\lfloor t/\delta \rfloor} G_C(x, f, r; \delta). \tag{A.6}$$

Since $\sup_C G_C(x, f, r; \delta) < \infty$, that for all for all $t > 0$, $M_t = \sup_C G_C(x, f, r; t) < \infty$. Let t_0 be such that for some $\gamma > 0$, $\inf_{x \in C} \mathbb{P}_x(\tau_B \leq t_0) \geq \gamma > 0$ (such constants always exist, see the proof of Proposition 3.4).

Let τ^k be the kth-iterate of $\tau = \tau_C(t_0)$

$$\tau^k = \tau^{k-1} + \tau \circ \theta^{\tau^{k-1}}, \qquad k \geq 2,$$

where θ is the usual shift operator. Define for $n \geq 2$, the $\{0, 1\}$-valued random variables $(u_n)_n$ by $u_n = 1$ iff $\tau_B \circ \theta^{\tau^{n-1}} \leq t_0$. Then by definition, $u_n \in \mathcal{F}_{\tau^n}$ and $\mathbb{P}_x (u_n = 1 | \mathcal{F}_{\tau^{n-1}}) \geq \gamma > 0$. Finally, set $\eta = \inf\{n \geq 2, u_n = 1\}$. Then it holds

$$G_B(x, f, r; t_0) \leq \mathbb{E}_x \left[\int_0^{\tau^\eta} r(s) f(X_s) \, ds \right] \leq \sum_{n \geq 2} \mathbb{E}_x \left[\int_0^{\tau^n} r(s) f(X_s) \, ds \; \mathbf{1}_{\eta \geq n} \right].$$

Define for all $n \geq 2$,

$$a_x(n) = \mathbb{E}_x \left[\int_0^{\tau^{n-1}} r(s) f(X_s) \, ds \; \mathbf{1}_{\eta \geq n} \right], \qquad b_x(n) = \mathbb{E}_x \left[r(\tau^{n-1}) \; \mathbf{1}_{\eta \geq n} \right].$$

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Then by the strong Markov property and the property $r(s + t) \leq r(s)r(t)$ for all $s, t \geq 0$, we have
$$G_B(x, f, r; t_0) \leq \sum_{n \geq 2} \left(a_x(n) + M_{t_0} b_x(n) \right).$$

Following the same lines as in the proof of [26, Lemma 3.1], it may be proved that for all $n \geq 3$
$$b_x(n) \leq \rho\, b_x(n - 1) + c\, (1 - \gamma)^{n-1}, \qquad a_x(n) \leq (1 - \gamma)\, a_x(n - 1) + M\, b_x(n - 1),$$

for some positive constants $c < \infty$ and $\rho < 1$. This proves that there exists a constant $c < \infty$ such that $G_B(x, f, r; t_0) \leq c\, G_C(x, f, r; t_0)$. By (A.6), there exists a constant c_{t_0} such that $G_B(x, f, r; t_0) \leq c_{t_0} G_C(x, f, r; \delta)$. This implies that $\sup_{x \in C} G_B(x, f, r; t_0) < \infty$. Finally, for all $n \geq 1$ we write

$$G_B(x, f, r; t_0 + n t_0) \leq \mathbb{E}_x \left[\int_0^{\tau_B(t_0) \circ \theta^{\tau_C^n(t_0)} + \tau_C^n(t_0)} r(s) f(X_s)\, ds \right]$$

$$\leq \mathbb{E}_x \left[\int_0^{\tau_C^n(t_0)} r(s) f(X_s)\, ds \right] + \mathbb{E}_x \left[r\left(\tau_C^n(t_0) \right) \mathbb{E}_{X_{\tau_C^n(t_0)}} \left[\int_0^{\tau_B(t_0)} r(s)\, f(X_s)\, ds \right] \right]$$

$$\leq \mathbb{E}_x \left[\int_0^{\tau_C^n(t_0)} r(s) f(X_s)\, ds \right] + \sup_{x \in C} G_B(x, f, r; t_0)\, \mathbb{E}_x \left[r\left(\tau_C^n(t_0) \right) \right].$$

Since $f \geq 1$ and $\lim_{t \to \infty} r(t) / \int_0^t r(s) ds = 0$ for all $r \in \Lambda_0$, there exists a constant $c < \infty$ such that for all n large enough

$$G_B(x, f, r; t_0 + n t_0) \leq c\, \mathbb{E}_x \left[\int_0^{\tau_C^n(t_0)} r(s) f(X_s)\, ds \right].$$

As in the proof of [11, Lemma 20] (see also [21, Lemma 4.1] for a similar calculation), the term in the right hand side is upper bounded by $c_{n t_0} G_C(x, f, r; \delta)$ and this concludes the proof.

A.4 Proof of Proposition 3.7

We prove that *(i)* and *(ii)* are equivalent. That *(ii)* implies *(i)* is trivial. For the converse implication, we start with proving that $\{x \in \mathsf{X}, G_C(x, f, r; \delta) < \infty\}$ is full. This can be done following the same lines as the proof of [21, Proposition 4.2] upon noting that (a) by [11, Lemma 20], there exists $M < \infty$ such that for all $t \geq 0$, $G_C(x, f, r; \delta + t) \leq G_C(x, f, r; \delta) + M^t$; (b) we can assume that C is ν_a-petite for some maximal irreducibility measure ν_a and a distribution a such that $\int M^t a(dt) < \infty$ ([21, Proposition 3.2(ii)]). Proposition 3.5 now implies that the sets $C_n = \{x \in \mathsf{X}, G_C(x, f, r; \delta) \leq n\}$ are (f, r)-regular and thus petite ([21, Proposition 4.2(i)]). As in the proof of Proposition 3.4, we thus deduce that there exists a (f, r)-regular set, which is petite, closed and accessible.

We have just proved that under *(i)*, the sets C_n are (f, r)-regular petite sets and $\cup_n C_n$ is full.

This shows that *(i)* $\Rightarrow$ *(iii)*.

We finally prove that *(iii)* $\Rightarrow$ *(ii)*. Since $\psi(\cup_n C_n) > 0$, $C_n \subset C_{n+1}$ and ψ is regular, there exists n_* and a compact set A such that $A \subseteq C_{n_*}$ and $\psi(A) > 0$. Hence, A is accessible; furthermore, it is (f, r)-regular (and thus petite) as a subset of a (f, r)-regular set.

A.5 Proof of Proposition 3.8

(i) We first prove that

$$\mathbb{E}_x \left[\sum_{k=0}^{T_{m,B} \wedge M} \phi \circ V(X_{mk}) \right] \leq m^{-1} \mathbb{E}_x \left[\int_0^{m(T_{m,B} \wedge M)} \{\phi \circ V(X_s) \, ds \right] + b\phi'(1) \, \mathbb{E}_x \left[m(T_{m,B} \wedge M) \right].$$
(A.7)

where M is any positive real number. Write

$$\mathbb{E}_x \left(\sum_{k=1}^{T_{m,B} \wedge M} \phi \circ V(X_{mk}) \right) - \mathbb{E}_x \left(\int_0^{T_{m,B} \wedge M} \phi \circ V(X_{ms}) ds \right)$$

$$= \mathbb{E}_x \left(\sum_{k=1}^{\infty} \left[\int_{k-1}^{k} \{\phi \circ V(X_{mk}) - \phi \circ V(X_{ms})\} ds \right] \mathbf{1}_{k \leq T_{m,B} \wedge M} \right)$$

$$\leq \mathbb{E}_x \left(\sum_{k=1}^{\infty} \left[\int_{k-1}^{k} \{\phi' \circ V(X_{ms})(V(X_{mk}) - V(X_{ms}))\} ds \right] \mathbf{1}_{k \leq T_{m,B} \wedge M} \right)$$

$$\leq \sum_{k=1}^{\infty} \int_{k-1}^{k} \mathbb{E}_x \left[\mathbb{E}_x \left(V(X_{mk}) - V(X_{ms}) | \mathcal{F}_{ms} \right) \phi' \circ V(X_{ms}) \mathbf{1}_{k \leq T_{m,B} \wedge M} \right] ds$$

$$\leq b\phi'(1)\mathbb{E}_x \left[\sum_{k=1}^{\infty} \int_{k-1}^{k} \int_{sm}^{km} \mathbf{1}_C(X_u) du \, ds \, \mathbf{1}_{k \leq T_{m,B} \wedge M} \right] = b\phi'(1)\mathbb{E}_x \left[\int_0^{m(T_{m,B} \wedge M)} \mathbf{1}_C(X_u) du \right]$$

$$\leq b\phi'(1)\mathbb{E}_x \left[m(T_{m,B} \wedge M) \right].$$

Finally,

$$\mathbb{E}_x \left[\int_0^{T_{m,B} \wedge M} \phi \circ V(X_{ms}) ds \right] = m^{-1} \, \mathbb{E}_x \left[\int_0^{m(T_{m,B} \wedge M)} \phi \circ V(X_s) ds \right],$$

and (A.7) is established. The drift condition $\mathbf{D}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b})$ and the optional sampling theorem imply

$$\mathbb{E}_x \left[\int_0^{m(T_{m,B} \wedge M)} \phi \circ V(X_s) ds \right] \leq V(x) + b \, \mathbb{E}_x \left[m(T_{m,B} \wedge M) \right].$$
(A.8)

Combining (A.7) and (A.8) yields

$$\mathbb{E}_x \left[\sum_{k=0}^{T_{m,B} \wedge M} \phi \circ V(X_{mk}) \right] \leq m^{-1} V(x) + c \, \mathbb{E}_x \left[T_{m,B} \wedge M \right],$$

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for some finite constant c. Since $\sup_C V < \infty$, by Proposition 3.4 and Theorem 3.1, there exist a closed accessible petite set A and for all $\delta > 0$, a finite constant c_δ such that

$$\mathbb{E}_x\left[\tau_A(\delta)\right] \leq c_\delta\, V(x), \qquad \sup_A V < \infty.$$

Furthermore, under the stated assumptions, the process is positive Harris-recurrent [21, Theorem 1.2] and since some skeleton is irreducible, there exists a maximal irreducibility measure ν and $t_0 > 0$ such that $\inf_{x \in A} \inf_{t \geq t_0} P^t(x, \cdot) \geq \nu(\cdot)$ ([23, Proposition 6.1] and [21, Proposition 3.2(ii)]). Hence, there exists $\gamma > 0$ such that $\inf_{x \in A} \inf_{t_0 \leq t \leq t_0 + m} \mathbb{P}_x\left(X_t \in B\right) \geq \gamma$. Following the same lines as in the proof of [11, Proposition 22(ii)], it may be proved that $\mathbb{E}_x\left[T_{m,B}\right] \leq c'V(x)$ for some constant $c' < \infty$, thus concluding the proof.

(ii) Since $r_* = \phi \circ H_\phi^{-1}$ is increasing,

$$\mathbb{E}_x\left[\sum_{k=0}^{T_{m,B}-1} r_*(km)\right] \leq \phi(1) + \mathbb{E}_x\left[\int_0^{mT_{m,B}} r_*(s)ds\right].$$

As in the previous case, we show that $\inf_{x \in A} \inf_{t_0 \leq t \leq t_0 + m} \mathbb{P}_x\left(X_t \in B\right) \geq \gamma > 0$ for some closed accessible petite set A. The result now follows from [11, Proposition 22(ii)] (with a minor modification : the authors claim that $T_{m,B} \leq \tau^\eta$ while we have $mT_{m,B} \leq \tau_A^\eta$) and Theorem 3.1.

A.6   Proof of Theorem 3.10

The theorem is a consequence of [11, Theorem 1] and of results by Tuominen and Tweedie [31] on discrete time Markov chains. We nevertheless have all the ingredients in this paper to rewrite the proof of [11, Theorem 1] in few lines. For ease of the proof of the new results, we start with this concise proof.

Let P^m be the irreducible skeleton. We can assume without loss of generality that $\Psi_1 \circ r_* \in \Lambda_0$, $\Psi_1 \circ r_* \geq 1$ and $\Psi_2 \circ f_* \geq 1$, and we do so. Write $t = km + u$ for some $0 \leq u < m$ and a non-negative integer k. Since $\Psi_1 \circ r_* \in \Lambda_0$ and is a non-decreasing rate function, $\Psi_1 \circ r_*(km + u) \leq \Psi_1 \circ r_*(km)\, \Psi_1 \circ r_*(m)$. Furthermore, if $|g| \leq \Psi_2 \circ f_*$, upon noting that Ψ_2 and ϕ are non-decreasing concave functions

$$P^u|g| \leq P^u(\Psi_2 \circ \phi \circ V) \leq \Psi_2 \circ \phi\left(P^u V\right) \leq \Psi_2 \circ \phi\left(V + bm\right) \leq \Psi_2(f_*) + mb\phi'(1) \leq c\,\Psi_2(f_*),$$

where we used that by (3.1), $P^u V \leq V + bu$. Hence, there exists a finite constant c such that

$$\Psi_1 \circ r_*(t)\, \|P^t(x, \cdot) - \pi(\cdot)\|_{\Psi_2 \circ f_*} \leq c\,\Psi_1 \circ r_*(km)\, \|P^{km}(x, \cdot) - \pi(\cdot)\|_{\Psi_2 \circ f_*}. \qquad \text{(A.9)}$$

By Proposition 3.4, there exists a V-level set $A = \{V \leq n\}$ which is accessible and petite for the process. Hence, under the stated assumptions, there exist t_0 and a maximal irreducibility measure ψ such that $\inf_{t \geq t_0} \inf_{x \in A} P^t(x, \cdot) \geq \psi(\cdot)$ ([23, Proposition 6.1] and [21, Proposition

3.2(ii)]). This implies that A is petite and accessible for the m-skeleton and P^m is aperiodic. Furthermore, by Proposition 3.8 and the inequality (3.3),

$$\sup_A \mathbb{E}_x \left[\sum_{j=0}^{T_{m,A}-1} \Psi_1 \circ r_*(jm) \ \Psi_2 \circ f_*(X_{jm}) \right] < \infty. \tag{A.10}$$

We now have all the ingredients to deduce (3.5) to (3.8) from known results on discrete-time Markov chains. Eq. (3.5) results from [31, Theorem 4.1, Eq(36)] while (3.6) is established in the proof of [31, Theorem 4.1]. (3.7) is a consequence of [31, Theorem 4.2]. Since $\partial[\Psi_1 \circ r_*] \in \Lambda_0$ (and thus is non-decreasing), there exists a finite constant c such that for all $0 \le |u| \le m$,

$$\partial[\Psi_1 \circ r_*](km + u) \le c\partial[\Psi_1 \circ r_*](km - u) \le cm^{-1} \int_{km-m}^{km} \partial[\Psi_1 \circ r_*](s) \ ds$$

$$\le cm^{-1}\{[\Psi_1 \circ r_*](km) - [\Psi_1 \circ r_*](km - m)\} = cm^{-1}\{\Delta[\Psi_1 \circ r_*](km)\},$$

where for a rate function r defined on the non-negative integers, we associate a sequence Δr defined by $\Delta r(0) = r(0)$ and $\Delta r(k) = r(k) - r(k-1)$, $k \ge 1$. Thus, there exists $c < \infty$ such that

$$\partial[\Psi_1 \circ r_*](t) \ \|P^t(x, \cdot) - \pi(\cdot)\|_{\Psi_2 \circ f_*} \le c \ \Delta[\Psi_1 \circ r_*](km) \ \|P^{km}(x, \cdot) - \pi(\cdot)\|_{\Psi_2 \circ f_*}.$$

Under the stated assumptions, $\{\Delta[\Psi_1 \circ r_*](km)\}_k$ is a subgeometric rate function defined on the integers (see e.g. the class Λ in [31]). Observe indeed that

$$\frac{\ln \partial[\Psi_1 \circ r_*](km)}{km} \le \frac{\ln \left(m^{-1} \int_{km}^{(k+1)m} \partial[\Psi_1 \circ r_*](s) \ ds\right)}{km} = \frac{\ln \left(m^{-1}\Delta[\Psi_1 \circ r_*](km + m)\right)}{km}$$

$$\le \frac{\ln \partial[\Psi_1 \circ r_*]((k+1)m) - \ln m}{(k+1)m} \frac{(k+1)m}{km}.$$

Since $\partial[\Psi_1 \circ r_*] \in \Lambda_0$, the discrete rate function $\{\Delta[\Psi_1 \circ r_*](km)\}_k$ is equivalent to the discrete rate function $\{\partial[\Psi_1 \circ r_*](km)\}_k$ which is in the class Λ_0 defined e.g. in [31]. (3.8) now follows from [31, Theorem 4.3].

A.7 Proof of Theorem 3.11

Since $V \in \mathcal{D}(\mathcal{A})$, there exists an increasing sequence $T_n \uparrow \infty$ of $\mathcal{F}_t$-stopping times such that for any n, $t \mapsto V(X_{t \wedge T_n}) - V(X_0) - \int_0^{t \wedge T_n} \mathcal{A}V(X_s)ds$ is a $\mathbb{P}_x$-martingale. Denote $a^+ = a \vee 0$. We have $(\mathcal{A}V)^+(x) \le b\mathbf{1}_C(x)$ and thus $\mathbb{E}_x(\int_0^{t \wedge T_n}(\mathcal{A}V)^+(X_s))ds < \infty$ which ensures that the quantity $\mathbb{E}_x(\int_0^{t \wedge T_n} \mathcal{A}V(X_s))ds$ is well defined. This implies that

$$0 \le \mathbb{E}_x(V(X_{t \wedge T_n})) = V(x) + \mathbb{E}_x \left(\int_0^{t \wedge T_n} \mathcal{A}V(X_s)ds \right) \le V(x) + b\mathbb{E}_x \left(\int_0^{t \wedge T_n} \mathbf{1}_C(X_s)ds \right) < \infty.$$

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This allows to write

$$\mathbb{E}_x(V(X_{t \wedge T_n})) + \mathbb{E}_x\left(\int_0^{t \wedge T_n} \phi \circ V(X_s)ds\right) = V(x) + \mathbb{E}_x\left(\int_0^{t \wedge T_n} [\mathcal{A}V(X_s) + \phi \circ V(X_s)]ds\right)$$

$$\leq V(x) + b\mathbb{E}_x\left(\int_0^{t \wedge T_n} \mathbf{1}_C(X_s)ds\right).$$

The previous inequality ensures in particular, by monotone convergence theorem, that $\mathbb{E}_x\left(\int_0^t \phi \circ V(X_s)ds\right) <$ ∞. The proof is now completed by noting that

$$\mathbb{E}_x(V(X_t)) = \mathbb{E}_x(\liminf_n V(X_{t \wedge T_n})) \leq \liminf_n \mathbb{E}_x(V(X_{t \wedge T_n}))$$

$$\leq \liminf_n \left\{V(x) - \mathbb{E}_x\left(\int_0^{t \wedge T_n} \phi \circ V(X_s)ds\right) + b\mathbb{E}_x\left(\int_0^{t \wedge T_n} \mathbf{1}_C(X_s)ds\right)\right\}$$

$$= V(x) - \mathbb{E}_x\left(\int_0^t \phi \circ V(X_s)ds\right) + b\mathbb{E}_x\left(\int_0^t \mathbf{1}_C(X_s)ds\right)$$

where the last equality follows from monotone convergence.

A.8 Proof of Theorem 3.12

We first prove (i). It is straightforward that since C is petite for the resolvent kernel, it is also petite for the Markov process associated to the semi group P_t. Now, by definition, we have

$$\mathbb{E}_x(R_\beta V(X_u)) = \int_0^\infty \beta e^{-\beta v} P^{v+u}(x, V)dv = e^{\beta u} R_\beta V(x) - e^{\beta u} \int_0^u \beta e^{-\beta v} P^v(x, V)dv . \quad \text{(A.11)}$$

This implies that

$$\mathbb{E}_x\left(\int_0^s \beta(R_\beta V(X_u) - V(X_u))du\right)$$

$$= \int_0^s \beta e^{\beta u} R_\beta V(x)du - \int_0^s \left(e^{\beta u}\int_0^u \beta e^{-\beta v} P^v(x, V)dv\right)du - \beta\int_0^s P^u(x, V)du$$

$$= (e^{\beta s} - 1)R_\beta V(x) - \int_0^s \left(\int_v^s \beta e^{\beta u}du\right) e^{-\beta v} P^v(x, V)dv - \beta\int_0^s P^u(x, V)du$$

$$= (e^{\beta s} - 1)R_\beta V(x) - e^{\beta s}\int_0^s \beta e^{-\beta v} P^v(x, V)dv = \mathbb{E}_x(R_\beta V(X_s)) - R_\beta V(x) \quad \text{(A.12)}$$

Moreover, if $\check{\mathbf{D}}(\mathbf{C}, \mathbf{V}, \phi, \mathbf{b}, \beta)$ holds then,

$$\mathbb{E}_x\left(\int_0^s \beta(R_\beta V(X_u) - V(X_u))du\right) \leq -\mathbb{E}_x\left[\int_0^s \beta\phi \circ V(X_u)du\right] + \beta b\mathbb{E}_x\left[\int_0^s \mathbf{1}_C(X_u)du\right] \quad \text{(A.13)}$$

Combining (A.12) and (A.13) yields (i). Now, consider (ii). By [21, Theorem 2.3 (i) and Proposition 4.4 (ii)] and Theorem 3.1, there exist positive constants δ, c_1 and c_2 such that for any $x \in \mathsf{X}$,

$$\breve{\mathbb{E}}_x \left[\sum_{k=1}^{\check{\tau}_C} \phi \circ V(\check{X}_k) \right] \leq G_C(x, \phi \circ V, \mathbf{1}; \delta) + c_1 \sup_{x \in C} G_C(x, \phi \circ V, \mathbf{1}; \delta) \leq V(x) + c_2$$

where $(\check{X}_k)_k$ is a Markov chain with transition kernel R_β, $\check{\tau}_C = \inf\{k \geq 1 : \check{X}_k \in C\}$ and $\breve{\mathbb{E}}_x$ is the expectation associated to $\breve{\mathbb{P}}_x$ the probability induced by the Markov chain $(\check{X}_k)_k$. Write $W(x) = \breve{\mathbb{E}}_x \left(\sum_{k=0}^{\check{\sigma}_C} \phi \circ V(\check{X}_k) \right)$ and fix $\epsilon > 0$ small enough so that $0 \leq \sup_{u \geq 1} \phi(u) - \epsilon u < \infty$. This implies that there exists some constant c such that

$$W(x) \leq (1 + \epsilon)V(x) + c, \qquad x \in \mathsf{X}.$$

Let $\check{C} = \{x \in \mathsf{X} : W(x) \leq \sup_C \phi \circ V + A\}$ where A is a positive number such that $(\sup_C \phi \circ V + A - c)/(1 + \epsilon) \geq 1$. Note that $C \subset \check{C}$ since if $x \in C$, $W(x) = \phi \circ V(x) \leq \sup_C \phi \circ V$ and thus, $x \in \check{C}$. This implies that for all $x \notin \check{C}$,

$$R_\beta W(x) = W(x) - \phi \circ V(x) \leq W(x) - \check{\phi} \circ W(x) , \qquad (A.14)$$

with $\check{\phi}$ is a non decreasing differentiable concave function such that $\check{\phi}(u) = \phi \left(\frac{u-c}{1+\epsilon} \right)$ for $u \geq \sup_C \phi \circ V + A$. Moreover, for all $x \in \check{C}$,

$$R_\beta W(x) - W(x) + \check{\phi} \circ W(x) \leq \sup_{\check{C}} \left\{ \breve{\mathbb{E}}_x \left[\sum_{k=1}^{\check{\tau}_C} \phi \circ V(\check{X}_k) \right] + \phi \circ V(x) \right\}$$

$$\leq \sup_{\check{C}} \left\{ V(x) + c_2 + \phi \circ V(x) \right\} . \quad (A.15)$$

Since $\phi \circ V \leq W$ on X, V and $\phi \circ V$ are finite on $\check{C}$. By (A.14) and (A.15), there exists a constant $\check{b}$ such that for all $x \in \mathsf{X}$,

$$R_\beta W \leq W - \check{\phi} \circ W + \check{b} \mathbf{1}_{\check{C}}.$$

Moreover, we have by straightforward algebra $\lim_t r_{\check{\phi}}(t)[r_\phi((1 + \epsilon)t)]^{-1} = 1 + \epsilon$. It remains to check that $\check{C}$ is petite w.r.t. R_β. Since $\check{C}$ is included in some set $\{V \leq n\}$ which is petite w.r.t. the semi group P_t, we have that $\check{C}$ is petite w.r.t. the semi group P_t which implies by [21, Proposition 3.2] that $\check{C}$ is petite w.r.t the Markov transition kernel R_β. The proof is completed.

A.9 Proof of Theorem 5.1

(i) We first prove that $P^m W \leq W - \phi \circ W + b' \mathbf{1}_C$. This a consequence of Proposition 3.8 and Theorem 14.2.3 (ii) in Meyn-Tweedie [22]. Indeed, since $\sup_C V < \infty$, *(i)* shows that

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$\sup_{x \in C} \mathbb{E}_x \left[\sum_{k=0}^{T_{m,C}-1} \phi \circ V(X_{km}) \right] < \infty$. Define $\sigma_{m,C} = \inf\{k \geq 0, X_{mk} \in C\}$ and set $W(x) = \mathbb{E}_x \left[\sum_{k=0}^{\sigma_{m,C}} \phi \circ V(X_{km}) \right]$. Then the function W satisfies the conditions (see [22, Chapter 14]). As discussed in the proof of Theorem 3.10, for all $n \geq n_*$ the level sets $\{V \leq n\}$ are accessible and petite for the skeleton chain P^m. As a consequence, either $\sup_C V \leq n_*$ and we may replace C by $\{V \leq n_*\}$ in the previous drift inequality, or $\sup_C V \geq n_*$ and we choose $\tilde{C} = C$.

(ii) The Moderate deviations principle (or MDP) comes from a decomposition into blocks and a return to the discrete time case. Assume that $m = 1$ which can be done without loss of generality. In fact, by *(i)*, the Markov chain $(\Xi_k := X_{[k,k+1[})_{k \in \mathbb{N}}$ with probability transition Q is subgeometrically ergodic with the invariant probability measure $\tilde{\pi} = \mathbb{P}_\pi|_{\mathcal{F}_1}$ and satisfies **A1-A2** in the terminology of Douc-Guillin-Moulines [5]. Then, we may write (denoting the integer part by $\lfloor \cdot \rfloor$)

$$
\begin{aligned}
S_t^\epsilon &= \frac{1}{\sqrt{\epsilon} h(\epsilon)} \int_0^t g(X_{s/\epsilon}) ds \\
&= \frac{\sqrt{\epsilon}}{h(\epsilon)} \int_0^{t/\epsilon} g(X_s) ds \\
&= \frac{\sqrt{\epsilon}}{h(\epsilon)} \sum_{k=0}^{\lfloor t/\epsilon \rfloor - 1} \int_k^{k+1} g(X_s) ds + \frac{\sqrt{\epsilon}}{h(\epsilon)} \int_{\lfloor t/\epsilon \rfloor}^{t/\epsilon} g(X_s) ds \\
&= \frac{\sqrt{\epsilon}}{h(\epsilon)} \sum_{k=0}^{\lfloor t/\epsilon \rfloor - 1} G(\Xi_k) + \frac{\sqrt{\epsilon}}{h(\epsilon)} \int_{\lfloor t/\epsilon \rfloor}^{t/\epsilon} g(X_s) ds
\end{aligned}
$$

where G is obviously a bounded mapping with values in $\mathbb{R}^n$. By the boundedness of g, it is easy to see that the second term is exponentially negligible in the sense of moderate deviations, and thus S_t^ϵ and $\frac{\sqrt{\epsilon}}{h(\epsilon)} \sum_{k=0}^{\lfloor t/\epsilon \rfloor - 1} G(\Xi_k)$ are exponentially equivalent, and share the same MDP.

Note now that by Theorem 7 of Douc-Guillin-Moulines [5], under the subgeometric ergodicity of (Ξ_k) and the condition on the speed, $\frac{\sqrt{\epsilon}}{h(\epsilon)} \sum_{k=0}^{\lfloor t/\epsilon \rfloor - 1} G(\Xi_k)$ satisfies a MDP with speed $\frac{1}{h^2(\epsilon)}$ and rate function

$$
\tilde{I}_g^h(\gamma) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 \sup_{\zeta \in \mathbb{R}^d} \left\{ \langle \dot{\gamma}(t), \zeta \rangle - \frac{1}{2} \tilde{\sigma}^2(\langle G, \zeta \rangle) \right\} dt & \text{if } d\gamma(t) = \dot{\gamma}(t) dt, \ \gamma(0) = 0, \\[4mm] +\infty & \text{else,} \end{cases}
$$

where

$$
\tilde{\sigma}^2(\langle G, \zeta \rangle) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\pi \left(\sum_{k=0}^{n-1} G(\Xi_k) \right)^2.
$$

On the other hand, by the subexponential ergodicity, the boundedness of g and $\mathbb{E}_\pi \langle g, \zeta \rangle = 0$,

we have that $\int_0^\infty (P_t \langle g, \zeta \rangle - \pi(\langle g, \zeta \rangle)) dt$ is absolutely convergent in $L^1(\pi)$. Thus

$$
\begin{aligned}
\tilde{\sigma}^2(\langle G, \zeta \rangle) &= \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_\pi \left(\int_0^n g(X_s) ds \right)^2 \\
&= \lim_{n \to \infty} \frac{2}{n} \mathbb{E}_\pi \left(\int_0^n ds \int_0^s \langle g, \zeta \rangle P^u \langle g, \zeta \rangle du \right) \\
&= 2 \int_{\mathsf{X}} \langle g, \zeta \rangle \int_0^\infty P^u \langle g, \zeta \rangle du \; d\pi \\
&= \sigma^2(\langle g, \zeta \rangle),
\end{aligned}
$$

and then $\tilde{I}_g^h = I_g^h$.

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