Simple estimate of signal to interference ratio with randomly located antennas

L. Decreusefond
GET - CNRS UMR 5141
Télécom Paris
46, rue Barrault
75 634 Paris cedex 13
Email: Laurent.Decreusefond@enst.fr

E. Ferreir
GET - CNRS UMR 5141
Télécom Paris
46, rue Barrault
75 634 Paris cedex 13
Email: eferreir@enst.fr

P. Martins
GET - CNRS UMR 5141
Télécom Paris
46, rue Barrault
75 634 Paris cedex 13
Email: Philippe.Martins@enst.fr

***Abstract*— We consider a propagation model under path loss, shadowing and multipath effects, where each cell are a Voronoi tessellation and the generating points of this cells are a uniform Poisson point process in $\mathbb{R}^2$. In addition, the antennas are located in each point generated by the Poisson process.**

With this model, we calculate the mean interference to signal ratio $\overline{\text{ISR}}$, which actually gives us a physical information, but once we are interest in apply this mean in the Shannon's formula, we simulated the real mean value of signal to interference ratio $\overline{\text{SIR}}$ and compared the result with the calculated $\overline{\text{ISR}}^{-1}$.

I. Introduction

In cellular radio systems, the most prominent feature is the coverage at each point. A point is covered if the signal to interference ratio is greater than a given threshold. The computation of this ratio depends on the positions of the antennas with respect to the users and a signal propagation model. Antennas positions are usually supposed to be the centers of an hexagonal lattice. We here investigate the situation where they are randomly located and compute the signal to interference ratio for a given customer. This paper does not aim to be as complete as [1] on the subject of stochastic geometric models for cellular radio systems. Our goal is to show that by a proper re-parametrization, one can simplify computations even in a random context.

Considering that the noise is much smaller than the total power interference caused by the other antennas, we can ignore the presence of power noise in a cell. We wish to find the expected value of signal to interference ratio (SIR) in a cell rounded by another cells. Let S be the useful signal and I be the interference. The actual value that we want to find is

$$\overline{\text{SIR}}_1 = \mathbf{E}\left[\frac{S}{I}\right].$$

However, $\overline{\text{SIR}}_1$ represents a somehow intractable integral. So, instead of $\overline{\text{SIR}}_1$, we will calculate the following mean:

$$\overline{\text{SIR}} = \frac{1}{\mathbf{E}\left[I/S\right]},$$

because it is analytically tractable and I/S gives the same physical information than S/I does. This does not mean that $\overline{\text{SIR}} = \overline{\text{SIR}}_1$. If we see the operation $\mathbf{E}[XY]$ as an inner product, we can use the Cauchy-Schwarz inequality to show that:

$$\begin{aligned} 1 = \mathbf{E}[1] &= \mathbf{E}\left[\frac{S}{I}\frac{I}{S}\right]^{1/2} \le \mathbf{E}\left[\frac{S}{I}\right]^{1/2}\mathbf{E}\left[\frac{I}{S}\right]^{1/2} \\ &\Rightarrow \overline{\text{SIR}}_1 \ge \overline{\text{SIR}}^{-1}. \end{aligned}$$

Thus $\overline{\text{SIR}}_1^{-1}$ gives a pessimistic, hence convenient for dimensioning purposes, evaluation of the signal to interference ratio.

The paper is organized as follows: we first proceed with some preliminaries which contain the re-parametrization trick, then we precise the model for wireless propagation we'll use. We then compute the mean value of the interference to signal ratio. We show by simulation, that the mean value of interference to signal ratio are within admissible bounds of the signal to interference ratio.

II. Preliminaries

A point process in a metric space X (here X will be $\mathbf{R}^+$ or $\mathbf{R}^2$) is a Poisson process of intensity measure λ when the following properties are satisfied (we refer to ([2]) for details): λ is a diffuse Radon measure; the number of points in a compact set B is distributed according to a Poisson random variable of parameter $\lambda(B)$; if A and B are two disjoints sets, the random variables $N(A)$ and $N(B)$ which count the number of points in A and B respectively, are independent.

For N a Poisson process of intensity measure λ on $X = \mathbf{R}^2$, we denote by $(X_n,\, n \ge 1)$ the locations of its atoms. The point process the atoms of which are $(\|X_n\|,\, n \ge 1)$, where $\|x\|$ is the euclidean norm of x, is an $\mathbf{R}^+$-valued Poisson process of intensity measure $\hat\lambda$ with $\hat\lambda([0,t]) = \lambda(B(0,\,t))$, where $B(0,\,t)$ is the ball of center 0 and radius t. Since $t \mapsto \hat\lambda([0,t])$ is increasing, it has a reciprocal function $\hat\lambda^{-1}$ such that $\hat\lambda(\lambda^{-1}([0,t])) = t$. It is a well-known result that the process V whose atoms are $(\hat\lambda^{-1}(\|X_n\|),\, n \ge 1)$ is a Poisson process of intensity 1. We denote by $(T_n,\, n \ge 1)$ the atoms of V in increasing order. It is then classical that the joint law of $(T_1,\, \cdots,\, T_n)$ has a density given by

$$e^{-t_n}\mathbf{1}_{\{0 \le t_1 \le \cdots \le t_n\}}. \tag{1}$$

In the following, we will need to consider only configurations for which T_1 is greater than a given threshold. We take this into account by considering the law of $(T_1, \cdots, T_n)$ conditionally to $T_1 > r_0$ whose density is given by

$$e^{r_0} e^{-t_n} \mathbf{1}_{\{0 \le t_1 \le \cdots \le t_n\}}.$$

III. Physical model

We use a standard model of propagation with path loss $P(r)$, shadowing g_s and multipath g_m (the last two are random gains with mean G_s and $G_m = 1$), combined together so that if P_r is the received power: $P_r = P(r) g_s g_m$. The path loss is usually considered as a deterministic variable, but if we take into account that the distance from the user to the antennas is a random variable, the path loss turns to a random variable too. Consider the coordinate system XOY, with a user in the origin O and antennas randomly placed in this plane. Moreover, we suppose that these antennas are the points of a Poisson point process with intensity λ, and each cell of each antenna is a Poisson Voronoi cell generated by the set of antenna locations. We consider the fixed referential as the user, so if he is moving with instantaneous velocity $\vec{v}$ in relation to the ground, in our coordinate system, all the antennas are moving with instantaneous velocity $-\vec{v}$ (this will be important to show that the movement of the user will not affect the mean value of the received power). Once the antennas are placed in a plane, the probability of two of them are contained in the same circumference centered in the origin is null. So, if the point configuration $X = X_1, X_2, \ldots$ is generated by the process, the distance of the i-th antenna to the user is given by $R_i(X)$. The one located at distance $R_1(X)$ is responsible for transmitting the useful signal $P_{r_1}(r) = P_1(r) g_{m_1} g_{s_1} = S$ and the one located at a distance $R_i(X)$ is responsible for the interference with power $P_{r_i}(r) = P_i(r) g_{m_i} g_{s_i}$, for $i = \overline{2, N}$ where N is the number is the number of antennas we take into consideration for the computation of the interferences. Since the physical phenomena that generate path loss, shadowing and multipath are completely distinct, we will consider g_{m_i}, g_{s_i} and P_{r_i} as independent. Moreover, there is no physical reason to believe that g_{s_i} or g_{m_i} depends on g_{s_j}, g_{m_j} or P_{r_j}, for $i, j = \overline{1, N}$, $i \ne j$, so this reasonable consideration will be done. For the sake of simplicity, we denote by R_i the euclidean norm of the ith closest point from the origin of the point process N. Then, the SIR will be given by:

$$\begin{aligned}
\overline{\text{SIR}}^{-1} &= \mathbf{E}\left[\frac{\sum_{i=2}^N P(R_i)}{P(R_1)}\right] = \mathbf{E}\left[\frac{\sum_{i=2}^N P(R_i) g_{m_i} g_{s_i}}{P(R_1) g_{m_1} g_{s_1}}\right] \\
&= \sum_{i=2}^N \mathbf{E}\left[\frac{P(R_i) g_{m_i} g_{s_i}}{P(R_1) g_{m_1} g_{s_1}}\right] \qquad (2) \\
&= \sum_{i=2}^N \mathbf{E}\left[\frac{P(R_i)}{P(R_1)}\right] \frac{\mathbf{E}[g_{m_i}]\mathbf{E}[g_{s_i}]}{\mathbf{E}[g_{m_1}]\mathbf{E}[g_{s_1}]} \\
&= \sum_{i=2}^N \mathbf{E}\left[\frac{P(R_i)}{P(R_1)}\right] \frac{G_s G_m}{G_s G_m} = \sum_{i=2}^N \mathbf{E}\left[\frac{P_i(r)}{P_1(r)}\right] \qquad (3)
\end{aligned}$$

This means that whatever the hypothesis made on shadowing and multipath, the result will hold as long as we consider these effects to be independent from path loss.

Theorem 1: Assume that the intensity measure λ is invariant with respect to translation in the plane (i.e., λ is constant times the Lebesgue measure on $\mathbf{R}^2$). If the cell related to an antenna is a Poisson Voronoi cell generated by the points where they are, and if handovers are done such that the nearest antenna is the one responsible for the useful signal, the mean value of any function of $(R_1(X), \ldots, R_n(X)),\ n \in \mathbb{N}$ does not depends on the user movement.

Proof. The theorem holds if $p_{R_1(X),\ldots,R_n(X)}(r_1, \ldots, r_n)$ does not change, so the mean of any function of $(R_1(X), \ldots, R_n(X))$ will be the same. We have to proof these two statements: (i) if a user moves without crossing a Poisson Voronoi edge the theorem holds; (ii) if a user crosses a Poisson Voronoi edge the theorem holds. Note that a translation $\vec{d}$ done by the user, in the user's referential is the same that a translation $-\vec{d}$ done by the antennas. Since, by hypothesis, the distribution of the process does not change under a translation, the first point holds. If the user crosses a Poisson Voronoi edge from a cell generated by X_i to one generated by X_j, doing a handover, $R_1(X)$ changes from $\|X_i\|$ to $\|X_j\|$ and $R_2(X)$ changes to $\|X_i\|$, and by invariance by translation, the distribution of $p_{R_1(X),\ldots,R_n(X)}(r_1, \ldots, r_n)$ does not change.

We are now interested in the computations of the last expectation in (3). The power is related to the radius by:

$$P(r) = P_t K \left[\frac{r_0}{r}\right]^\gamma = \frac{D^\gamma}{r^\gamma} \text{for } r > r_0 > 0, \qquad (4)$$

where $D = (P_t K)^{1/\gamma} r_0$ and, under some circumstances, if f_0 is the frequency of the carrier and c is the speed of light, $K = c/4\pi d_0 f_0$. Thus, we have

$$\mathbf{E}\left[\frac{\sum_{i=2}^N P(R_i)}{P(R_1)} \mid R_1(X) \ge r_0\right] = \sum_{i=2}^N e^{r_0} \int e^{-t_n} \hat{\lambda}^{-1}(t_i)^{-\gamma} \hat{\lambda}^{-1}(t_1)^{\gamma} \mathbf{1}_{r_0 \le t_1 \le \ldots \le t_n} \, dt_1 \ldots dt_n. \qquad (5)$$

From now on, we choose λ to be a constant c times the Lebesgue measure on $\mathbf{R}^2$, we then have $\hat{\lambda}(t) = \sqrt{t/c\pi}$. It follows from (5) that $\overline{\text{SIR}}^{-1}$ does not depend on c, i.e., does not depend on the number on antennas per square-meter as long as they are "uniformly" located. We computed these integrals by a recurrence formula given below and proved in the appendix. The sequence of functions, $H_n(\alpha)$, is given by

$$H_n(\alpha) = \begin{cases} 1, & \text{if } n = 0; \\ \prod_{m=1}^n (\alpha + m), & \text{if } n > 0. \end{cases} \qquad (6)$$

The sequence of functions, $F_n(\alpha)$, is given by

$$F_n(\alpha) = \frac{\Gamma(n + \alpha + 1, r_0)}{H_n(\alpha)} - \sum_{j=1}^n \frac{r_0^{j+\alpha}}{H_j(\alpha)} \left\{ \sum_{m=0}^{n-j} \frac{(-r_0)^{n-j-m} \Gamma(m+1)}{(n-j-m)!m!} \right\}$$

The value of $\overline{\text{SIR}}$ is given by

$$\overline{\text{SIR}} = \frac{1}{\sum_{i=2}^{N} \overline{\text{ISR}}_i},$$

where $\overline{\text{ISR}}_i$ is defined as

$$\overline{\text{ISR}}_i = e^{r_0} \left[\frac{F_{(N-i)}(i-1)}{H_{i-1}(\gamma/2)} - \sum_{j=1}^{i-1} \frac{r_0^{j+\gamma/2}}{H_j(\gamma/2)} \sum_{m=1}^{i-j-1} \frac{(-r_0)^{i-j-m-1}}{(i-j-m-1)!m!} F_{(N-i)}(m-\gamma/2) \right].$$

We then compare these exact computations to simulation based values of $\overline{\text{SIR}}_1$. Figure 1 that for different numbers of interfering antennas, the difference between the easily computed quantity and the usually accounted value, is not that bad.

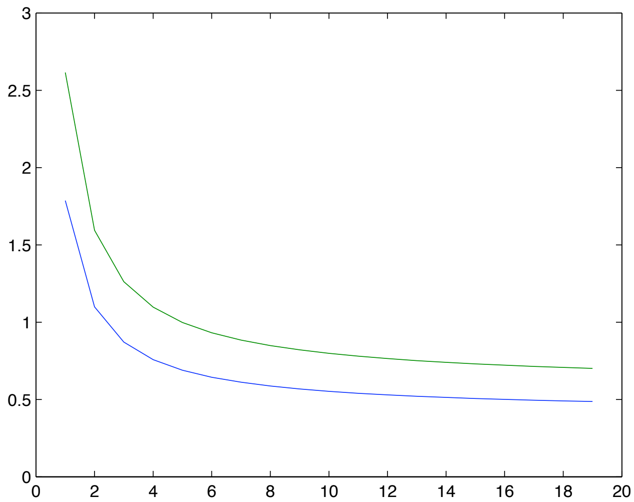

Fig. 1. Comparison between $\overline{\text{SIR}}$ and $\overline{\text{SIR}}_1$.

Appendix

Considering only the ith term in the sum, we can write the mean interference to signal ratio $\overline{\text{ISR}}_i$ as:

$$\overline{\text{ISR}}_i = e^{(a_0)} \underbrace{\int_{a_0}^{\infty} e^{-a_N} \int_{a_0}^{a_{i+1}} a_i^{\gamma/2} \int_{a_0}^{a_j} \int_{a_0}^{a_2}}_{N \text{ integrals}} a_1^{-\gamma/2}\, da_1 \dots da_N. \tag{7}$$

To solve this integral, we will define two useful sequences. The first one is $V_n(a_i)$:

$$V_n(a_i) = \begin{cases} 1, \text{ if } n = 0; \\ \displaystyle\int_{a_0}^{a_i} \int_{a_0}^{a_n} \dots \int_{a_0}^{a_3} \int_{a_0}^{a_2} da_1 da_2 \dots da_{n-1} da_n, \\ \quad \text{if } n \geq 1. \end{cases} \tag{8}$$

By induction, we have

$$V_n(a_i) = \frac{(a_i - a_0)^n}{n!}. \tag{9}$$

The second one is given by

$$f_n(\alpha, a_i) = \begin{cases} a_1^{\alpha}, \\ \text{if } n = 0; \\ \displaystyle\int_{a_0}^{a_i} \int_{a_0}^{a_n} \dots \int_{a_0}^{a_3} \int_{a_0}^{a_2} a_1^{\alpha} da_1 \dots da_n, \\ \quad \text{if } n \geq 1. \end{cases} \tag{10}$$

The recurrence relation of this sequence is:

$$f_n(\alpha, a_i) = \frac{f_{n-1}(\alpha+1, a_i)}{\alpha+1} - \frac{a_0^{\alpha+1} V_{n-1}(a_i)}{\alpha+1}, \tag{11}$$

so we can write $f_n(\alpha, a_i)$ using equation (11) as

$$f_n(\alpha, a_i) = \frac{a_i^{n+\alpha}}{H_n(\alpha)} - \sum_{j=1}^{n} \frac{a_0^{j+\alpha} V_{n-j}(a_i)}{H_j(\alpha)}. \tag{12}$$

Now we can evaluate equation (7) until we are left with $N-i+1$ integrals:

$$\begin{aligned} \overline{\text{ISR}}_i &= e^{a_0} \underbrace{\int_{a_0}^{\infty} e^{-a_N} \dots \int_{a_0}^{a_{i+1}}}_{N-i+1 \text{ integrals}} a_i^{\gamma/2} \\ &\qquad f_{i-1}(\gamma/2, a_i)\, da_i \dots da_N \\ &= e^{a_0} \underbrace{\int_{a_0}^{\infty} e^{-a_N} \dots \int_{a_0}^{a_{i+1}}}_{N-i+1 \text{ integrals}} \left(\frac{a_i^{i-1}}{H_{i-1}(\gamma/2)} \right. \\ &\qquad \left. - a_i^{-\gamma/2} \sum_{j=1}^{i-1} \frac{a_0^{j+\gamma/2} V_{i-1-j}(a_i)}{H_j(\gamma/2)} \right) da_i \dots da_N, \end{aligned} \tag{13}$$

and writing in terms of the defined functions:

$$\begin{aligned} \overline{\text{ISR}}_i = e^{(a_0)} \int_{a_0}^{\infty} e^{-a_N} &\left[\frac{f_{(N-i)}(i-1, a_N)}{H_{N-i}(\gamma/2)} \right. \\ - \sum_{j=1}^{i-1} \frac{a_0^{j+\gamma/2}}{H_j(\gamma/2)} &\left(\sum_{m=1}^{i-j-1} \frac{(-a_0)^{i-j-m-1}}{(i-j-m-1)!m!} \right. \\ &\left. \left. f_{(N-i)}(m-\gamma/2, a_N) \right) \right] da_N. \end{aligned} \tag{14}$$

At this point, we have already reduced the N-tuple integral to a simple integral of sums. All these integrals can be evaluated using the incomplete Gamma function which has the form

$$\int_{a_0}^{\infty} r a_N^q e^{-a_N} da_n = \frac{r\Gamma(q+1, a_0)}{(q+1)}, \tag{15}$$

q and r real constants. Once

$$F_n(\alpha) = \int_{a_0}^{\infty} e^{-a_N} f_n(\alpha, a_N) da_N,$$

we substitute this in 14 and obtain the final desired expression.

REFERENCES

[1] F. Baccelli, B. Blaszczyszyn, and F. Tournois, "Spatial averages of coverage characteristics in large CDMA networks," *Wirel. Netw.*, vol. 8, no. 6, pp. 569–586, 2002.

[2] A. F. Karr, *Point processes and their statistical inference*, 2nd ed., ser. Probability: Pure and Applied. New York: Marcel Dekker Inc., 1991, vol. 7.