MAXIMUM DISTANCE BETWEEN SLATER ORDERS AND COPELAND ORDERS OF TOURNAMENTS

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Abstract – Given a tournament $T = (X, A)$, we consider two tournament solutions applied to T: Slater's solution and Copeland's solution. Slater's solution consists in determining the linear orders obtained by reversing a minimum number of directed edges of T in order to make T transitive. Copeland's solution applied to T ranks the vertices of T according to their decreasing out-degrees. The aim of this paper is to compare the results provided by these two methods: to which extent can they lead to different orders? We consider three cases: T is any tournament, T is strongly connected, T has only one Slater order. For each one of these three cases, we specify the maximum of the symmetric difference distance between Slater orders and Copeland orders. More precisely, thanks to a result dealing with arc-disjoint circuits in circular tournaments, we show that this maximum is equal to $n(n - 1)/2$ if T is any tournament on an odd number n of vertices, to $(n^2 - 3n + 2)/2$ if T is any tournament on an even number n of vertices, to $n(n - 1)/2$ if T is strongly connected with an odd number n of vertices, to $(n^2 - 3n - 2)/2$ if T is strongly connected with an even number n of vertices greater than or equal to 8, to $(n^2 - 5n + 6)/2$ if T has an odd number n of vertices and only one Slater order, to $(n^2 - 5n + 8)/2$ if T has an even number n of vertices and only one Slater order.

Keywords – Majority tournament; tournament solutions; Slater orders; Slater winners; Copeland orders; Copeland winners; symmetric difference distance; arc-disjoint circuits in circular tournaments.

1. INTRODUCTION, DEFINITIONS AND NOTATION

1.2. Introduction

At the end of the eighteenth century (for the historic context, see [8], [32], [33], [34] and [35]), M. J. A. N. Caritat, marquis de Condorcet, studied a problem arising in voting theory: the aggregation of linear orders into a linear order [9]. To solve this problem, he suggested to apply a pairwise comparison method.

To describe such a method, let X be a finite set of *candidates*; n will denote the number of candidates (i.e., $n = |X|$). We consider that m voters are asked to rank the elements of X. Condorcet's method consists in computing, for each candidate $x \in X$ and each candidate

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$y \in X$ with $x \neq y$, the number m_{xy} of voters who prefer x to y. The *(strict) majority relation* is the relation T defined by: $xTy \Leftrightarrow m_{xy} > m_{yx}$ (for a more general presentation on this topic, see [5]). If there is no tie, what will be assumed in the sequel, T is then a *tournament* (called the *majority tournament of the election*), i.e. a complete asymmetric relation: for any pair of candidates $\{x, y\}$ with $x \neq y$, one and only one of the following two situations occurs: xTy or yTx. From the graph theoretic point of view, a tournament $T = (X, A)$ is a directed, complete, asymmetric graph: between two distinct vertices x and y ($x \neq y$), there exists one and only one of the two arcs (i.e., directed edges) (x, y) or (y, x) (for references on tournaments, see [30], [36], [39], [40], as well as [31] for a catalogue of non isomorphic tournaments with at most 10 vertices or for some families of tournaments; more generally, see for instance [2] or [6] for the bases of graph theory). It is well-known that a tournament is transitive if and only if it is without any circuit (i.e., directed cycle) and, in this case, it is a linear order. It is also well-known that a tournament T is strongly connected if and only if there exists a Hamiltonian circuit, i.e. a circuit going through each vertex of T exactly once.

Notice that, even if we assume the preferences of the voters to be linear orders defined over X, the majority tournament T is not necessarily a linear order, because T may not be transitive: a candidate x can be preferred to another candidate y by a majority of voters, y to a third candidate z by another majority of voters, and z to x by a third majority of voters. Such a situation, discovered by Condorcet himself, is known as the « voting paradox » or also as the « effet Condorcet » in French (see [19]). But T can also be a linear order. A linear order O defined on X will be represented by a permutation $(x_1, x_2, \ldots, x_n)$ of the elements of $X = \{1, 2, ..., n\}$. In such a representation of O, a candidate x_i will be considered as preferred to another candidate x_j according to O if x_i occurs before x_j in the writing of the permutation i.e., for the permutation considered above $(x_1, x_2, \ldots, x_n)$, if i is smaller than j. The candidate x_1 who is ranked in first position of the considered order O will be called the *winner* of O. More generally, if a tournament T admits a vertex x fulfilling the following property: $\forall\, y \in X$, xTy, then x is the *winner* of T. With respect to the election summarized by T, the winner of T, if any, is the *Condorcet winner* of the election, i.e. a candidate preferred to all the others by a majority of voters. If a Condorcet winner exists, there is only one.

The following example gives an illustration of a majority tournament.

Example 1. Consider $X = \{1, 2, 3, 4\}$ and $m = 16$. The preferences of the 16 voters are assumed to be the following, with the notation described above for representing the linear orders:

- (1, 2, 3, 4) for 4 voters
- (3, 4, 1, 2) for 2 voters
- (4, 1, 3, 2) for 5 voters
- (2, 3, 4, 1) for 5 voters.

The majority tournament associated with this election is the tournament of Figure 1.

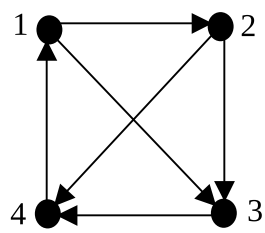

FIGURE 1. The majority tournament of Example 1.

When the considered tournament is not a linear order, we can wonder how to rank the candidates and which candidate(s) must be or can be considered as the winner(s) of the election. Different answers can be brought to this question, known under the name of *tournament solutions* (see for instance [27], [30] or [37]). In this paper, we pay attention to two tournament solutions (of which the definitions are specified below): the solution designed by P. Slater [42] and the one by A.H. Copeland [16] (notice that, according to [33], Ramon Llull (ca 1232–1316) promoted the method of pairwise comparison and proposed the Copeland rule to select a winner; despite this historical discovery, we shall keep the usual authorships). These two methods give the possibility to construct linear orders to rank the candidates and to define winners of the election, called Slater winners or Copeland winners, from the winners of these orders. The question that we study in this paper is the following: to which extend can the rankings provided by these two methods be different?

Thanks to an example with 7 vertices, J.-C. Bermond [7] already showed that the Copeland winners and the Slater winners can define disjoint sets. In fact, such a situation can occur for any $n \geq 6$ (see [12] or [14]). More precisely, these two sets are equal for $n \leq 3$, the set of Copeland winners contains the one of Slater winners for $n = 4$, and the intersection of the two sets is non-empty for $n = 5$ but there is no systematic inclusion between them. The relationships between Slater's solution or Copeland's solution on the one hand and other tournament solutions on the other have already been investigated (see [12]): it is the case for instance for the solution of J.G. Kemeny [22] (see [13] and [23]), for the solution of J. Banks [3] (see [15], [20], [25], [38]), for the solution of C.L. Dodgson (also known as Lewis Carroll) [17] (see [24]), or for the prudent orders [1] (see [28]).

Last, notice that the maximum likelihood method proposed by E. Zermelo [45] (see also [30]) yields to the same winners and to the same rankings as Copeland's solution. Consequently, all the results below between Copeland's solution and Slater's solution can also be applied for a comparison between Zermelo's solution and Slater's solution.

1.2. Definitions and notation

In the sequel, T will denote a tournament of order n. The vertices of T will be 1, 2, …, n. By definition of a tournament, for any pair $\{x, y\}$ with $1 \leq x \leq n$, $1 \leq y \leq n$ and $x \neq y$, there exists exactly one of the two arcs (i.e., directed edge) (x, y) or (y, x). If x and y are two vertices of T such that the arc (x, y) exists, we say that *x beats y* and that *y is beaten by x*.

A transitive tournament is a linear order and conversely. A tournament is transitive if and only if it is without circuit (i.e. directed cycle). If two circuits do not share any arc in common, they are said to be *arc-disjoint* (they may share a common vertex). If $x_1, x_2, \ldots, x_n$ is a permutation of the vertices of T, we say that *T is the transitive tournament defined by the order* $(x_1, x_2, \ldots, x_n)$ if, for any pair $\{i, j\}$ with $1 \leq i \leq n$, $1 \leq j \leq n$ and $i \neq j$, x_i beats x_j. The *reversed order* of an order $(x_1, x_2, \ldots, x_n)$ is the order $(x_n, x_{n-1}, \ldots, x_1)$.

If $(x_1, x_2, \ldots, x_n)$ is a permutation of the vertices of T, we say that T is a *circular tournament* defined by the order $(x_1, x_2, \ldots, x_n)$ if, for any pair $\{i, j\}$ with $1 \leq i < j \leq n$, x_i beats x_j if we have $j - i \leq \left\lfloor \frac{n}{2} \right\rfloor$ (otherwise, x_j beats x_i). We may notice that, for n odd, all the vertices of a circular tournament play the same role (more precisely, for any given i and j, there exists an automorphism of the circular tournament such that the image of x_i is x_j). Figure 2 represents, on the left, the circular tournament of order 5 defined by the order (1, 2, 3, 4, 5) and, on the right, the circular tournament of order 6 defined by the order (3, 4, 5, 6, 1, 2).

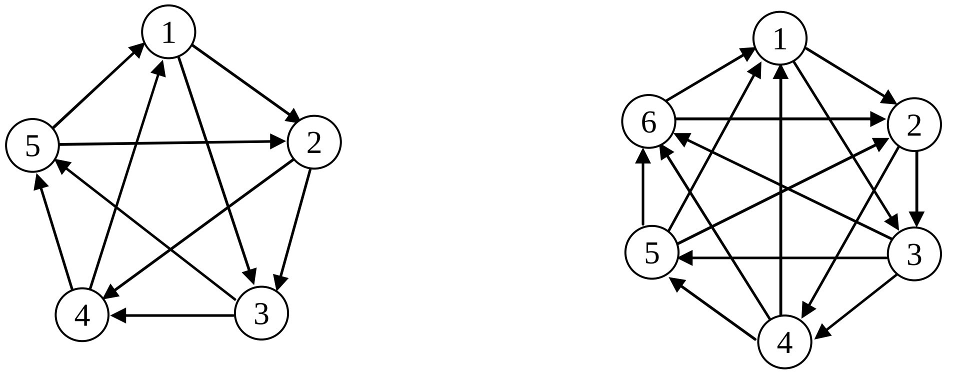

FIGURE 2. A circular tournament of order 5 (on the left) and a circular tournament of order 6 (on the right).

The (Copeland) *score* $s(x)$ of a vertex x of a tournament T is the number of vertices beaten by x, also called the *out-degree* of x. A *Copeland order* of a tournament T is any linear order obtained by sorting the vertices of T according to their non-increasing scores. A *Copeland winner* of T is the winner of a Copeland order of T. In other words, a Copeland winner of T is a vertex with a maximum score. For instance, the tournament of Example 1 admits four Copeland orders: (1, 2, 3, 4), (2, 1, 3, 4), (1, 2, 4, 3), (2, 1, 4, 3), which involves that 1 and 2 are the Copeland winners of this tournament. H.G. Landau [29] gave a characterization of the scores of a tournament:

THEOREM 1. Let $s_1, s_2, ..., s_n$ be n integers with $0 \le s_1 \le s_2 \le ... \le s_n$. These integers can be the scores of a tournament if and only if the following two properties are fulfilled:

1. $\forall\, 1 \le i \le n-1,\ \sum_{j=1}^{i} s_j \ge \frac{i(i-1)}{2}$;
2. $\sum_{j=1}^{n} s_j = \frac{n(n-1)}{2}$.

A tournament is said to be *regular* if n is odd and if all its vertices have a same score, thus equal to $(n-1)/2$. It is the case for instance for the circular tournaments with an odd number of vertices. Notice that, for n even, there does not exist regular tournament. For the circular tournaments with n even, there are $n/2$ vertices with a score equal to $n/2$ and $n/2$ vertices with a score equal to $(n-2)/2$.

Let T and T' be two tournaments (which can be linear orders) with the same set of vertices, the symmetric difference distance (see [4] for its axiomatic properties and [5] for its uses in the social sciences) between T and T' is the number of pairs $\{i, j\}$ with $1 \le i \le n$, $1 \le j \le n$ and $i \ne j$ for which the arc between i and j has not the same direction in T and in T'. This distance will be noted $dist(T, T')$. This distance is always less than or equal to $\frac{n(n-1)}{2}$. We call *concordance* between T and T', and we note $conc(T, T')$, the difference between this maximum and the distance between T and T': $conc(T, T') = \frac{n(n-1)}{2} - dist(T, T')$. In this paper, we study concordances between orders; the distance between these orders can easily be deduced from the concordances.

Let T be a tournament and O a linear order with the same vertices as T. An order O is a *Slater order of T* if, for any linear order O' defined on the vertices of T, we have:

$dist(T, O) \le dist(T, O')$. So, a Slater order of T is defined as an order at minimum distance from T. We call *Slater index* of T, and we note $i(T)$, the distance from T to any Slater order of T. This index can be interpreted as the minimum number of arcs of T which must be reversed in T to obtain a linear order (see [12] for equivalent formulations and for references on Slater's problem). A *Slater winner* of T is the winner of a Slater order of T. It is easy to see that the tournament of Example 1, which is not a linear order (for example the circuit (1, 2, 4) prevents this tournament from being transitive), becomes transitive by reversing the arc (4, 1). It is also easy to check that the only way to make this tournament transitive by the reversing of only one arc consists precisely in reversing the arc (4, 1). Thus, the Slater index of this tournament is equal to 1, with only one Slater order: (1, 2, 3, 4); so, 1 is its only Slater winner.

The computation of the Slater index of a tournament T, of the Slater orders of T, or of the Slater winners of T is NP-hard (see [21]). There exist anyway some tournaments for which these quantities are known. It is the case for instance for the circular tournaments (see [43]):

THEOREM 2. Let C_n be the circular tournament on n vertices defined by $(1, 2, ..., n - 1, n)$. Then we have:

- if n is odd, $i(C_n) = \frac{n^2 - 1}{8}$, there are n Slater orders of C_n which are $(1, 2, ..., n)$, $(2, 3, ..., n, 1)$, ..., $(n, 1, 2, ..., n - 1)$, and all the vertices are Slater winners;
- if n is even, $i(C_n) = \frac{n^2 - 2n}{8}$, there is only one Slater order which is $(1, 2, ..., n)$, and 1 is the only Slater winner.

More generally, it is easy to show that, for a regular tournament (thus, with n odd), all the vertices are simultaneously Copeland and Slater winners (see Lemma 3 below). A tournament solution which systematically selects all the vertices of a regular tournament as its winners is sometimes said to be *regular* (see for instance [30]); thus Slater's and Copeland's solutions are regular.

Notice that the so-called *Copeland value of a tournament solution* is based on Copeland solution (see [26] or [30]). Let *Sol* be a tournament solution and, for any tournament $T = (X, A)$, let $Sol(T)$ be the set of the winners of T according to *Sol*. The Copeland value VC_{Sol} of *Sol* for T is defined as the ratio $VC_{Sol}(T) = \frac{\max\{s(x) \text{ for } x \in Sol(T)\}}{\max\{s(x) \text{ for } x \in X\}}$. We obviously get the bounds $0 \le VC_{Sol}(T) \le 1$ for any tournament solution *Sol* and any tournament T. The Copeland value of the tournament solution *Sol* is defined as $\inf\{VC_{Sol}(T) : T \in \mathcal{T}\}$, where $\mathcal{T}$ denotes the set of all the tournaments; this value is also between 0 and 1 for any tournament solution. For Slater's solution, the Copeland value is equal to 0.5 (see [30]). In other words, the score of a Slater winner of a tournament T is between the maximum score of the vertices of T and half this maximum score, and we can get as close as desired to these two values.

1.3. Contribution of the study

The aim of this paper is to compute, for any given integer $n \ge 3$, the maximum distance or, equivalently, the minimum concordance between a Slater order and a Copeland order of a same tournament T with n vertices. In more formal terms, we want to compute the following quantity, for any $n \ge 3$:

$$\max\{dist(O_S(T), O_C(T)): T \text{ is any tournament on } n \text{ vertices, } O_S(T) \text{ is any Slater order of } T \text{ and } O_C(T) \text{ is any Copeland order of } T\},$$

or, equivalently:

$$\min\{conc(O_S(T), O_C(T)): T \text{ is any tournament on } n \text{ vertices, } O_S(T) \text{ is any Slater order of } T \text{ and } O_C(T) \text{ is any Copeland order of } T\}.$$

In Section 2, we compute this maximum distance when T is any tournament or, thanks to a decomposition of the set of the arcs of a circular tournament with an odd number of vertices into arc-disjoint circuits, when T is strongly connected. Section 3 is devoted to the case for which T has only one Slater order. The results are summarized in Section 4, which is the conclusion. In order to make the paper more reader-friendly, the proof of a theorem of Section 3 (Theorem 9) has been moved in an Appendix (Section 5).

Notice that the study for which there would be only one Copeland order is uninteresting: from the characterization provided by H.G. Landau [29], it appears that the uniqueness of the Copeland order can be observable only for transitive tournaments; in this case, there is also only one Slater order, which is the same as the Copeland order, i.e. the tournament itself. For this reason, we assume in the sequel that n is greater than or equal to 3.

The following lemmas will be useful in the sequel (see [7], [10], [11], [41]) :

LEMMA 3. Let $O_S = (x_1, x_2, \ldots, x_n)$ be a Slater order of T. For any i between 1 and n, x_i beats at least half the vertices $x_{i+1}, \ldots, x_n$ and at most half the vertices $x_1, x_2, \ldots, x_{i-1}$. If x_i beats exactly half the vertices $x_{i+1}, \ldots, x_n$, the order $(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_n, x_i)$ is also a Slater order of T; similarly, if x_i beats exactly half the vertices $x_1, x_2, \ldots, x_{i-1}$, then the order $(x_i, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_n)$ is also a Slater order of T.

Proof. Let α be the number of vertices among $x_{i+1}, \ldots, x_n$ beaten by x_i and $\beta = n - i - \alpha$ the number of vertices among $x_{i+1}, \ldots, x_n$ who beat x_i. Consider the order $O = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_{i+2}, \ldots, x_n, x_i)$. We get: $dist(T, O) - dist(T, O_S) = \alpha - \beta$. As O_S is assumed to be a Slater order, we obtain $\alpha - \beta \geq 0$, or also $\alpha - (n - i - \alpha) \geq 0$. Hence $\alpha \geq \frac{n-i}{2}$: x_i beats at least half the vertices $x_{i+1}, \ldots, x_n$. Moreover, if we have $\alpha = \beta$, then we get $dist(T, O) = dist(T, O_S)$ and O is also a Slater order.

We can prove the results with respect to the vertices $x_1, x_2, \ldots, x_{i-1}$ in a similar way.♦

COROLLARY 4. Let $O_S = (x_1, x_2, \ldots, x_n)$ be a Slater order of T. The score of x_1 is at least equal to $(n-1)/2$ and the one of x_n is at most equal to $(n-1)/2$.

Proof. Apply Lemma 3 with $i = 1$ or $i = n$. ♦

The result of Corollary 4, already used by J.-C. Bermond [7], was generalized by A. Guénoche [18] who designed a way to compute bounds of the ranks that a vertex x may occupy in a Slater order according to the score of x.

The following lemma (see [41]) shows that each Slater order $(x_1, x_2, \ldots, x_n)$ of T induces a Hamiltonian path, namely the Hamiltonian path made of the arcs (x_i, x_{i+1}) for $1 \leq i < n$ (notice that Lemma 5 shows a certain similarity to what is called Condorcet property for preference functions in [44]).

LEMMA 5. Let $O_S = (x_1, x_2, \ldots, x_n)$ be a Slater order of a tournament T. For each integer i between 1 and $n - 1$, x_i beats x_{i+1}.

Proof. If the statement of Lemma 5 was wrong, we would obtain a better order by switching x_i and x_{i+1} in O_S, a contradiction with the optimality of O_S. ♦

2. CASE WHERE *T* IS ANY TOURNAMENT OR IS STRONGLY CONNECTED

2.1. Case where T is any tournament

Let T denote any tournament. We want to show that, for any given odd $n \geq 3$, the minimum concordance between a Slater order and a Copeland order of a tournament with n vertices is equal to 0 while, for any given even $n \geq 4$, this minimum concordance is equal to $n - 1$. Theorem 6 provides a slightly stronger result.

THEOREM 6. Let n be an integer greater than or equal to 3.
1. If n is odd, there exists a tournament T on n vertices such that, for any Slater order O_S of T, there exists a Copeland order O_C with $conc(O_S, O_C) = 0$.
2. If n is even, let T be a tournament on n vertices and let O_S be any Slater order of T and O_C any Copeland order of T. Then $conc(O_S, O_C) \geq n - 1$. Moreover, there exists a tournament T such that, for any Slater order of T, there exists a Copeland order with $conc(O_S, O_C) = n - 1$.

Proof.
1. For n odd, any regular tournament T allows to conclude, since then any linear order is a Copeland order, in particular the reversed orders of the Slater orders of T.
2. Assume now that n is even and consider a tournament T of order n, a Slater order O_S of T and a Copeland order O_C of T. Let $(x_1, x_2, \ldots, x_n)$ be the Slater order. A vertex is said to be *large* if its score is at least equal to $\frac{n}{2}$ and is said to be *small* otherwise.

According to Corollary 4, x_1 is large and x_n is small. Let n_l denote the number of large vertices and n_s the number of small vertices. The vertex x_1 is ranked in the same way with respect to the small vertices in O_S and in O_C; so x_1 brings n_s to the value of the concordance. Similarly, x_n is ranked in the same way with respect to the large vertices in O_S and in O_C; if we do not consider x_1 anymore (x_1 is already considered above), x_n adds $n_l - 1$ to the concordance. Hence: $conc(O_S, O_C) \geq n_s + n_l - 1$. As $n_s + n_l$ is equal to n, we get the relation $conc(O_S, O_C) \geq n - 1$ for any tournament T, any Slater order O_S and any Copeland order O_C of T.

Let us show now that there exist tournaments with $conc(O_S, O_C) = n - 1$ for any even n. For this, consider the tournament T obtained from the circular tournament on $n - 1$ vertices defined by the order $(1, 2, \ldots, n - 1)$ by adding the vertex n in such a way that n is beaten by all the other vertices. Figure 3 represents such a tournament for $n = 6$. Then the order $(1, 2, \ldots, n - 2, n - 1, n)$ is a Slater order and the order $(n - 1, n - 2, \ldots, 2, 1, n)$ is a Copeland order of the constructed tournament. The concordance between these two orders is equal to $n - 1$. ♦

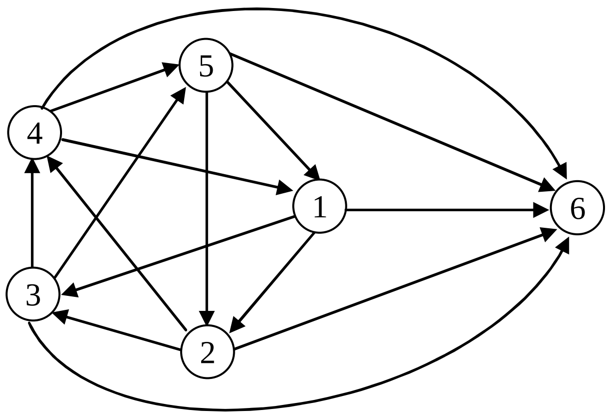

FIGURE 3. A tournament with $conc(O_S, O_C) = n - 1$ for $n = 6$.

2.2. Decomposition of a circular tournament with an odd order

In a directed graph, we call *arc-disjoint circuits* any set of circuits such that any two circuits of this set have no arc in common. We call *arc-decomposition of a tournament T into arc-disjoint circuits*, or simply a *decomposition of the arcs of T*, any set Z of arc-disjoint circuits such that any arc of T belongs to one and only one circuit of Z. Such a decomposition will be used in the sequel to study the distance between Slater orders and Copeland orders of tournaments.

THEOREM 7. Let T be a circular tournament with an odd number n of vertices. The number of arc-disjoint circuits in T is at most equal to $\frac{(n-1)(n+1)}{8}$. Any set of arc-disjoint circuits reaching this bound is a decomposition of the arcs of T into arc-disjoint circuits.

Proof.

We set $n = 2p + 1$ and we assume, without loss of generality, that T is the circular tournament defined by the order $(1, 2, 3, ..., n)$.

Let $\mathcal{C} = (c_1, c_2, \ldots, c_q)$ be any circuit of T. We first show that there exists a vertex x_1 of $\mathcal{C}$ with $1 \le x_1 \le p + 1$ and a vertex x_2 of $\mathcal{C}$ with $p + 2 \le x_2 \le 2p + 1$. We suppose that the vertices of $\mathcal{C}$ are numbered in such a way that c_1 is the smallest integer; T owns the arc (c_q, c_1) and so (c_1, c_q) is not an arc of T. As T is the circular tournament defined by the order (1, 2, 3, .., n), we have $c_q - c_1 \ge p + 1$. From $c_1 \ge 1$, we deduce the inequality $c_q \ge p + 2$ and, from $c_q \le 2p + 1$, we obtain $c_1 \le p$. So any circuit of $\mathcal{C}$ owns an arc (c_q, c_1) with $1 \le c_1 \le p + 1$ and $p + 2 \le c_q \le 2p + 1$ and thus, conversely, $\mathcal{C}$ must own an arc (x_1, x_2) with x_1 between 1 and $p + 1$ and x_2 between $p + 2$ and $2p + 1$.

Thus, the number of arc-disjoint circuits of T is at most equal to the number of arcs of which the head is between 1 and $p + 1$ and the tail is between $p + 2$ and $2p + 1$. If we consider the vertex i with $1 \le i \le p + 1$, i beats the p vertices $i + 1, i + 2, ..., i + p$. Among them, there are $p + 1 - i$ vertices lower than or equal to $p + 1$ and so $i - 1$ vertices between $p + 2$ and $2p + 1$. The number of arcs with their heads between 1 and $p + 1$ and their tails between $p + 2$ and $2p + 1$ is hence equal to $\sum_{i=1}^{p+1}(i-1) = \frac{p(p+1)}{2} = \frac{(n-1)(n+1)}{8}$.

Suppose now that there are exactly $\frac{p(p+1)}{2}$ arc-disjoint circuits. Then, the previous proof shows that all the arcs with their heads between 1 and $p+1$ and their tails between $p+2$ and $2p+1$ are involved in these circuits. Let now (i, j) be any arc of T. We can perform a circular permutation on the numbers of all the vertices of T so that i takes the number $p+1$ (remember that all the vertices play the same role); in this case, T is still defined by the order $(1, 2, 3, .., n)$ with respect to the new numbering of the vertices. Then the new number of j is between $p+2$ and $2p+1$. The result obtained previously shows that the arc (i, j) belongs to one of the considered arc-disjoint circuits. This completes the proof of Theorem 7. ♦

For $n = 3$, the upper bound of Theorem 7 is equal to 1 and is clearly reached. For $n = 5$, this upper bound is equal to 3 and is reached for instance by the decomposition given by the following three arc-disjoint circuits: (1, 2, 4), (2, 3, 5), (1, 3, 4, 5). If $n = 7$, the upper bound is equal to 6 and is reached for instance by the decomposition given by the following six arc-disjoint circuits: (1, 2, 3, 4, 5, 6), (1, 4, 7), (3, 6, 7), (2, 5, 7), (2, 4, 6), (1, 3, 5). For $n \geq 9$, we prove that the upper bound is reached in Theorem 8, with a sharper result, which will be used in the proof of Theorem 9.

THEOREM 8. Let T be a circular tournament on an odd number n of vertices with $n \geq 9$, defined by the order $(1, 2, 3, .., n)$. There exists a decomposition of the arcs of T into $\frac{(n-1)(n+1)}{8}$ arc-disjoint circuits with the circuit $(1, 2, 3, \ldots, n)$ as one of them, the others circuits being of length 3 or 4.

Proof.

We set $n = 2p + 1$ and we prove the result by induction on p. Notice the equality: $\frac{(n-1)(n+1)}{8} = \frac{p(p+1)}{2}$.

For $p = 4$, we have the desired decomposition, given by the 10 circuits (1, 2, 3, 4, 5, 6, 7, 8, 9), (1, 3, 7), (1, 4, 6), (1, 5, 8), (2, 4, 8), (2, 5, 7), (2, 6, 9), (3, 5, 9), (3, 6, 8), (4, 7, 9).

Let $p \geq 5$. We assume that the result is true for $p-1$ and we prove it for p. Let T be the circular tournament with $2p+1$ vertices and defined by the order $(1, 2, 3, ..., 2p+1)$. It is easy to check that the subgraph of T obtained by removing the two vertices $p+1$ and $2p+1$ is also a circular tournament T', defined by the order $(1, 2, \ldots, p, p+2, p+3, \ldots, 2p-1, 2p)$. According to the induction hypothesis, there exist, in T', $\frac{(p-1)p}{2} - 1$ circuits which are arc-disjoint and which do not use, for $i \in \{1, 2, \ldots, p-1, p+2, p+3, \ldots, 2p-1\}$, the arcs $(i, i+1)$, nor the arc $(p, p+2)$, nor the arc $(2p, 1)$. In addition to these circuits, consider the following extra circuits of T:

- for $i \in \{2, 3, \ldots, p-2\}$, the circuit $(2p+1, i, p+1, 2p-i+1)$;
- the circuit $(1, p+1, 2p)$;
- the circuit $(p-1, p+1, 2p+1)$;
- the circuit $(p, p+2, 2p+1)$.

Thus we obtain $\frac{(p-1)p}{2} - 1 + (p-3) + 3 = \frac{p(p+1)}{2} - 1$ arc-disjoint circuits which do not use, for any i between 1 and $2p$, the arcs $(i, i+1)$ nor the arc $(2p+1, 1)$. Hence the statement of the theorem, by adding the Hamiltonian circuit $(1, 2, \ldots, 2p, 2p+1)$ to the previous circuits. ♦

2.3. Case where T is strongly connected

We consider now the distance, or rather the concordance, between a Slater order and a Copeland order of a strongly connected tournament.

Let T_6 denote the tournament on 6 vertices of Figure 4. It is easy to check that the Slater index of T_6 is equal to 4 and that (1, 2, 3, 4, 5, 6) is one of its Slater orders. On the other hand, the order (5, 3, 1, 6, 4, 2) is a Copeland order of T_6. The concordance between these two orders is equal to 6.

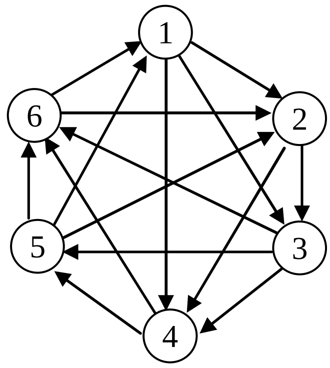

FIGURE 4. The tournament T_6

THEOREM 9. Let n be a positive integer greater than or equal to 3.
1. If n is odd, there exists a strongly connected tournament T on n vertices such that, for any Slater order O_S of T, there exists a Copeland order O_C of T with $conc(O_S, O_C) = 0$.
2. If n is equal to 4, the only strongly connected tournament on 4 vertices is the circular tournament, and the minimum concordance between its unique Slater order and its Copeland orders is 4.
3. If n is equal to 6, the minimum concordance, over the set of strongly connected tournaments T on 6 vertices, between the Slater orders of T and the Copeland orders of T is equal to 6; this minimum can be reached only for the tournament T_6 of Figure 4.
4. If n is even and is greater than or equal to 8, the minimum concordance, over the set of strongly connected tournaments T on n vertices, between the Slater orders of T and the Copeland orders of T is equal to $n + 1$.

In order to make the paper more reader-friendly, the proof of Theorem 9, rather long, has been moved to the Appendix.

3. CASE WHERE *T* HAS ONLY ONE SLATER ORDER

The last result deals with the minimum concordance between Slater orders and Copeland orders when the considered tournament has only one Slater order. As noticed above, the case where there is only one Copeland order is the one for which the tournament is transitive; then the tournament itself is the only Slater order and simultaneously the only Copeland order.

THEOREM 10. Let T be a tournament with $n \geq 3$ vertices and with only one Slater order. Let O_S be the Slater order of T and O_C be a Copeland order of T. If n is even, we have $conc(O_S, O_C) \geq 2n - 4$ and the bound can be tight. If n is odd, we have $conc(O_S, O_C) \geq 2n - 3$ and the bound can be tight.

Proof.

Assume that n is even and set $n = 2p$, with $p \geq 2$.

Let T be a tournament on n vertices with a unique Slater order $O_S = (x_1, x_2, \ldots, x_n)$. Let O_C be a Copeland order of T. Let us show the inequality $conc(O_S, O_C) \geq 2n - 4$. We use the terminology and notation used in the proofs of Theorems 6 and 9. We may show, as for these two theorems, that x_1 is large and that x_n is small. Assume now that x_2 is small; its score is exactly equal to $p - 1$: otherwise, we could move x_2 behind x_n to obtain a better order, and O_S would not be a Slater order; in the set $\{x_3, x_4, \ldots, x_n\}$, the number of vertices beaten by x_2 is equal to the number of vertices which beat x_2; so, the order $(x_1, x_3, x_4, \ldots, x_n, x_2)$ is also a Slater order, a contradiction with the uniqueness of the Slater order. Thus x_2 is large. We may prove in a similar way that x_{n-1} is small.

As shown in the proof of Theorem 6, the part of the concordance between O_S and O_C due to x_1 and x_n is equal to $n - 1$. We may now add the contribution due to x_2 with respect to the vertices other than x_1 and x_n; this contribution is at least equal to $n_s - 1$. Similarly, the contribution due to x_{n-1} with respect to the vertices other than x_1, x_2 and x_n is at least equal to $n_l - 2$. We get $conc(O_S, O_C) \geq n - 1 + n_s - 1 + n_l - 2 = 2n - 4$.

To prove that this lower bound is reached, consider the tournament T defined as follows. Let T_{circ} be the circular tournament defined by the order $(1, 2, \ldots, 2p - 1)$ and add the vertex $2p$ as well as the arcs $(i, 2p)$ for $i \in \{1, 2, \ldots, 2p - 2\}$ and the arc $(2p, 2p - 1)$. Let K be the Slater index of a circular tournament on $2p - 1$ vertices. Let O be any order. The distance between T and O is the sum of the distance between the restriction O' of O to the set $\{1, 2, \ldots, 2p - 1\}$ from T_{circ}, which is at least equal to K, and the contribution due to the vertex $2p$, which is at least 0. So the distance between T and O is at least K. To reach a distance exactly equal to K, we must have simultaneously that:

- O' is a Slater order of T_{circ};
- the vertices $1, 2, \ldots, 2p - 2$ are just before the vertex $2p$ in O and that the vertex $2p - 1$ is just after; by transitivity, the vertices $1, 2, \ldots, 2p - 2$ must be before the vertex $2p - 1$ in O.

The only Slater order of T_{circ} for which the vertices $1, 2, \ldots, 2p - 2$ are before $2p - 1$ is the order $(1, 2, \ldots, 2p - 1)$ (see Theorem 2). The only order O which is at distance K from T is thus the order $(1, 2, \ldots, 2p - 2, 2p, 2p - 1)$. Hence the uniqueness of the Slater order of T.

The scores of the vertices $1, 2, \ldots, 2p - 2$ are p, the one of the vertex $2p - 1$ is $p - 1$ and the one of the vertex $2p$ is 1. So the order $(2p - 2, 2p - 3, \ldots, 2, 1, 2p - 1, 2p)$ is a Copeland order and it is easy to check that the concordance between this order and the only Slater order is equal to $2n - 4$.

Now, let n be an odd integer and set $n = 2p + 1$.

Let T be a tournament on n vertices with only one Slater order $O_S = (x_1, x_2, \ldots, x_n)$. Let O_C be any Copeland order of T. We want to show that we have $conc(O_S, O_C) \geq 2n - 3$.

If $n = 3$, T must be the transitive tournament and the concordance is equal to 3, i.e. $2n - 3$.

Assume now that we have $n \geq 5$.

Let us define three kinds of vertices:

- the vertices of which the scores are greater than p; these vertices will be said to be *large*;

- the vertices of which the scores are equal to p; these vertices will be said to be *average*;
- the vertices of which the scores are less than p; these vertices will be said to be *small*.

Let n_l be the number of large vertices, n_a the number of average vertices, and n_s the number of small vertices; we have: $n_l + n_a + n_s = n$. Because of Corollary 4, the score of x_1 is at least $\frac{n-1}{2}$, i.e. p; if the score was exactly p, then the order $(x_2, x_3, \ldots, x_n, x_1)$ would also be a Slater order; since T has only one Slater order, x_1 is large; similarly, x_n is small. Because of Lemma 3, the score of x_2 is at least $\frac{n-2}{2}$, and so at least equal to p: x_2 is average or large; similarly, x_{n-1} is average or small.

The concordance due to x_1 is then at least $n_a + n_s$, the concordance due to x_n without respect to x_1 is at least $n_a + n_l - 1$. We now distinguish between four cases.

1. Vertices x_2 and x_{n-1} are average. The concordance due to x_2 without respect to x_1 and x_n is at least $n_s - 1$; the concordance due to x_{n-1} without respect to x_1, x_2 and x_n is at least $n_l - 1$. So the whole concordance is at least:
$$(n_a + n_s) + (n_a + n_l - 1) + (n_s - 1) + (n_l - 1) = 2n - 3.$$
2. Vertex x_2 is average and vertex x_{n-1} is small. The concordance due to x_2 without respect to x_1 and x_n is at least $n_s - 1$. The concordance due to x_{n-1} without respect to x_1, x_2 and x_n is at least $n_l - 1 + n_a - 1$. So the whole concordance is at least:
$$(n_a + n_s) + (n_a + n_l - 1) + (n_s - 1) + (n_l + n_a - 2) = 2n + n_a - 4.$$
As x_2 is average, we have $n_a \geq 1$ and thus the concordance is at least $2n - 3$.
3. Vertex x_2 is large and vertex x_{n-1} is average. This case can be dealt with as the previous one and we keep the same conclusion: the concordance is at least $2n - 3$.
4. Vertex x_2 is large and vertex x_{n-1} is small. The concordance due to x_2 without respect to x_1 and x_n is at least $n_s - 1 + n_a$. The concordance due x_{n-1} without respect to x_1, x_2 and x_n is at least $n_l - 2 + n_a$. So the whole concordance is at least:
$$(n_a + n_s) + (n_a + n_l - 1) + (n_s + n_a - 1) + (n_l + n_a - 2) = 2n + 2\,n_a - 4.$$
If n_a is not equal to 0, the concordance is greater than $2n - 3$. If n_a is equal to 0, consider x_3. Its score is greater than $\frac{n-3}{2}$, i.e. at least equal to p: otherwise O_S would not be a Slater order or would not be the only one. So, x_3 must be large or average. As n_a is assumed to be equal to 0, x_3 is large. By the same way, we may show that x_{n-2} is small. Thus, we have necessarily $n \geq 7$ and the couple (x_3, x_{n-2}) adds 1 to the concordance. So the whole concordance is at least $2n - 3$.

To show that the bound is tight for n odd, consider the tournament T defined as follows. We start from a circular tournament T_{circ} on $2p + 1$ vertices defined by the order $(1, 2, \ldots, 2p + 1)$ in which we reverse the arc $(2p + 1, 1)$ in order to obtain the arc $(1, 2p + 1)$. Let K denote the Slater index of T_{circ}. Let O be any order defined on $\{1, 2, \ldots, 2p, 2p + 1\}$. The distances from O to T_{circ} and to T differ by 1. More precisely, we have: $dist(O, T) = dist(O, T_{circ}) - 1$ if and only if vertex 1 is located in O before vertex $2p + 1$.

By definition of the Slater index of a tournament, we have $dist(O, T_{circ}) \geq K$, from which we obtain: $dist(T, O) \geq K - 1$. Moreover, the equality $dist(O, T) = K - 1$ is reached if and only if:

- $dist(O, T_{circ}) = K$, which means that O is a Slater order of T_{circ};
- in O, vertex 1 is before vertex $2p + 1$.

There is only one Slater order of T_{circ} in which vertex 1 is before vertex $2p + 1$ (see Theorem 2): it is the order $(1, 2, \ldots, 2p, 2p + 1)$. This order is thus the only Slater order of T.

On the other hand, $(1, 2p, 2p - 1, \ldots, 2, 2p + 1)$ is a Copeland order of T. The concordance between this order and $(1, 2, \ldots, 2p, 2p + 1)$ is equal to $2n - 3$.

This completes the proof of Theorem 10. ♦

4. CONCLUSION

The previous results are summarized below, stated in terms of distances. More precisely, we give below the maximum of the distance between the Slater orders and the Copeland orders of a tournament T with $n \geq 3$ vertices, over three sets of tournaments T: the set of any tournaments on n vertices, the set of strongly connected tournaments with n vertices, the set of tournaments on n vertices with only one Slater order.

maximum distance	any tournaments (Th. 6)	strongly connected tournaments (Th. 9)	tournaments with only one Slater order (Th. 10)
n odd	$n(n-1)/2$	$n(n-1)/2$	$(n^2 - 5n + 6)/2$
n even	$(n^2 - 3n + 2)/2$	2 if $n = 4$ 9 if $n = 6$ $(n^2 - 3n - 2)/2$ if $n \geq 8$	$(n^2 - 5n + 8)/2$

5. APPENDIX: PROOF OF THEOREM 9.

For n odd, the circular tournaments allow to conclude once again, as in the first step of the proof of Theorem 6.

Assume now that n is even. We consider a strongly connected tournament T defined on n vertices, a Slater order O_S of T, a Copeland order O_C of T and we try to minimize $conc(O_S, O_C)$.

If $n = 4$, as said above, the only (up to an isomorphism) strongly connected tournament on 4 vertices is the circular tournament of Figure 1. Then (1, 2, 3, 4) is the only Slater order of this tournament, and the most different Copeland order is the order (2, 1, 4, 3), for which the concordance is equal to 4.

We now assume that n is greater than or equal to 6, and we suppose that we have $conc(O_S, O_C) \leq n$; then we want to show, through the next nine steps, that then n is equal to 6 and that T is the tournament T_6.

To do this, we consider once again the notation of the proof of Theorem 6, but with T strongly connected. In particular, $(x_1, x_2, \ldots, x_n)$ denotes a Slater order O_S and we say that a vertex is *large* if its score is greater than or equal to $\frac{n}{2}$, and *small* otherwise. As in the proof of Theorem 6, vertex x_1 is large and vertex x_n is small.

* *Step 1*: x_1 cannot be the only large vertex and x_n cannot be the only small vertex.

Indeed, assume that x_1 is the only large vertex. Then the sum of the scores of the vertices $x_2, \ldots, x_{n-1}, x_n$ is at most equal to $(n-1)\left(\frac{n}{2}-1\right)$. As the sum of all the scores is equal to $\frac{n(n-1)}{2}$ according to Theorem 1, we obtain $\frac{n(n-1)}{2} - (n-1)\left(\frac{n}{2}-1\right) = n - 1$ as the score of x_n: x_n beats all the other vertices and hence T is not strongly connected, a contradiction. Thus there exist large vertices other than x_1. Similarly, we can prove that x_n is not the only small vertex.

* *Step 2*: if x_2 is large, then x_3 is small. Similarly, if x_{n-1} is small, then x_{n-2} is large.

Assume that x_1, x_2 and x_3 are large. Then x_1, x_2 and x_3 provide a concordance with respect to the small vertices equal to $3n_s$ and x_n provides an extra concordance with respect to the large vertices other than x_1, x_2 and x_3 equal to $n_l - 3$. The whole concordance is at least equal to $3n_s + n_l - 3$, and so to $n + 2n_s - 3$, since $n_s + n_l$ is equal to n. As we suppose that we have $conc(O_S, O_C) \leq n$, we obtain $2n_p - 3 \leq 0$ and so $n_s \leq 1$, a contradiction with the result of Step 1. Hence the first part of the statement of Step 2.

The second part of Step 2 can be showed in a similar way.

* *Step 3*: x_2 is small and x_{n-1} is large.

Assume that x_2 is large.

By considering the part of concordance provided by x_1 and x_2 with respect to the small vertices and the one provided by x_n with respect to the large vertices other than x_1 and x_2, we obtain a whole concordance at least equal to $2n_s + n_l - 2$, i.e. to $n + n_s - 2$, since we still have $n_s + n_l = n$. Hence $n_s \leq 2$. Thanks to the previous steps, we obtain the equality $n_s = 2$ and the fact that x_3 and x_n are the only small vertices. Moreover, the scores of the small (respectively large) vertices in O_S are increasing (respectively decreasing): otherwise the concordance would be greater than n.

According to Lemma 3, we have $s(x_3) \geq \frac{n-3}{2}$; as n is even and as x_3 is small, we have exactly $s(x_3) = \frac{n}{2}-1$. From $s(x_n) \geq s(x_3)$ and from the fact that x_n is small, we get $s(x_n) = \frac{n}{2}-1$.

Then the sum of the scores is at least equal to $(n-2)\frac{n}{2}+2(\frac{n}{2}-1)$, i.e. to $\frac{n^2}{2}-2$. But $\frac{n^2}{2}-2$ is greater than $\frac{n(n-1)}{2}$ for $n > 4$, a contradiction with Theorem 1. So x_2 must be small.

We prove similarly that x_{n-1} is large.

* *Step 4*: $s(x_2) = \frac{n}{2}-1$, $s(x_{n-1}) = \frac{n}{2}$.

Since x_2 is small, we have $s(x_2) \leq \frac{n}{2} - 1$. On the other hand, as O_S is a Slater order, Lemma 3 shows that x_2 beats at least half the vertices $x_3, x_4, \ldots, x_{n-1}, x_n$, which involves the inequality $s(x_2) \geq \frac{n-2}{2}$. Hence: $s(x_2) = \frac{n}{2}-1$.

The equality $s(x_{n-1}) = \frac{n}{2}$ can be shown in a similar way.

**Step 5*: $s(x_1) = \frac{n}{2}$ or $s(x_n) = \frac{n}{2}-1$.

The concordance brought by x_1 with respect to the small vertices and by x_n with respect to the large vertices is equal to $n_s + n_l - 1 = n - 1$. Assume that we have $s(x_1) > \frac{n}{2}$; since $s(x_{n-1})$ is equal to $\frac{n}{2}$, we must add 1 to the computation of the concordance for the vertices x_1 and x_{n-1}. Similarly, if we would have $s(x_n) < \frac{n}{2}-1$, it would be necessary to add 1 once again for the vertices x_2 and x_n, since the score of x_2 is equal to $\frac{n}{2}-1$, which would lead to a concordance at least equal to $n + 1$, a contradiction with the hypothesis. In conclusion, $s(x_1) > \frac{n}{2}$ involves $s(x_n) = \frac{n}{2}-1$. Hence the statement of Step 5.

* *Step 6*: if x_3 is small, then $s(x_1) > \frac{n}{2}$; if x_{n-2} is large, then $s(x_n) < \frac{n}{2} - 1$.

According to Lemma 5, x_1 beats x_2. As $s(x_2)$ is equal to $\frac{n}{2}-1$, x_2 beats exactly half the vertices $x_3, x_4, \ldots, x_{n-1}, x_n$: so, to move it inside O_S to put it at the last position gives another Slater order, namely $(x_1, x_3, x_4, \ldots, x_n, x_2)$. Then, according to Lemma 5, x_1 also beats x_3.

Assume now that x_3 is small. By applying Step 4 to the Slater order (x_1, x_3, x_4, …, x_n, x_2), we prove the equality $s(x_3) = \frac{n}{2} - 1$. Moving now x_3 to put it at the last position provides a third Slater order $O = (x_1, x_4, x_5, \ldots, x_n, x_2, x_3)$. The vertex x_1 beats at least half the vertices x_4, x_5, …, x_n: otherwise, we would obtain an order closer to T than O_S by moving x_1 inside O to put it between x_n and x_2. So we have $s(x_1) \geq 2 + \frac{n-3}{2} = \frac{n+1}{2}$, and then $s(x_1) > \frac{n}{2}$. Similarly, if x_{n-2} is supposed to be large, then we obtain $s(x_n) < \frac{n}{2} - 1$.

* *Step 7*: x_3 is large or x_{n-2} is small.

This follows from the previous two steps.

* *Step 8*: x_3 is large and x_{n-2} is small.

Assume that x_3 and x_{n-2} are small. According to Step 6, we have $s(x_1) > \frac{n}{2}$ and, according to Step 4, $s(x_{n-1}) = \frac{n}{2}$. With respect to the previous computations of the concordance, because of the vertices x_1 and x_{n-1}, we may add 1 to the concordance $n_s + n_l - 1$ brought by x_1 with respect to the small vertices and by x_n with respect to the large vertices other than x_1. We thus obtain a contribution to the concordance equal to n. As we have $s(x_2) = \frac{n}{2} - 1$, all the small vertices have a score equal to $\frac{n}{2} - 1$: otherwise, the concordance would increase because of x_2 with respect to such a small vertex. If n is greater than 6, let i be an integer with $4 \leq i \leq n - 3$. As the concordance is already equal to n, the pair (x_i, x_{n-2}) must not bring any extra concordance, which involves that x_i is small. Thus there are exactly two large vertices: x_1 and x_{n-1}, with scores equal to $\frac{n}{2}$, and the other vertices have a score equal to $\frac{n}{2} - 1$. In this case, as n is not equal to 4, the characterization of the scores (Theorem 1) is not satisfied.

So, x_3 et x_{n-2} cannot be small simultaneously. But Step 7 shows that, if x_3 is small, then x_{n-2} is small too. This involves that x_3 is large.

We may prove that x_{n-2} is small in a similar way.

* *Step 9*: $n = 6$ and T is isomorphic to T_6.

We now know that x_1, x_3 and x_{n-1} are large while x_2, x_{n-2} and x_n are small.

The contribution to the concordance brought by x_1 with respect to the small vertices and by x_n with respect to the large vertices is equal to $n_s + n_l - 1 = n - 1$. To this, we must add 1 for the contribution brought by (x_3, x_{n-2}), hence a total of n. Assume that we have $n > 6$; then: $4 < n - 2$. If x_4 is large, we must add 1 once again in the computation of the concordance for (x_4, x_{n-2}) and, if x_4 is small, we must also add 1 for (x_3, x_4), what is impossible since we assumed the equality $conc(O_S, O_C) = n$. So, in order to obtain $conc(O_S, O_C) = n$, we must have:

- $n = 6$,
- vertices x_1, x_3 and x_5 have a score equal to 3,
- vertices x_2, x_4 and x_6 must have a score equal to 2.

Because of Lemma 5, for $i \in \{1, 2, 3, 4, 5\}$, x_i beats x_{i+1}. Moreover, x_2 beats half the vertices x_3, x_4, x_5, x_6 and thus can be moved behind x_6, which involves that x_1 beats x_3 and that x_6 beats x_2. Similarly, by moving x_5 in front of x_1, we show that x_6 is beaten by x_4 and x_1 is beaten by x_5. As the score of x_3 is 3 and since x_3 is beaten by x_1 and x_2, then x_3 beats x_4, x_5 and x_6. As the score of x_4 is 2 and since x_4 beats x_5 and x_6, then x_4 is beaten by x_1, x_2 and x_3. As the score of x_2 is 2 and since x_2 beats x_3 and x_4, then x_2 is beaten by x_1, x_5 and x_6. Last, as the score of x_1 is 3 and since x_1 beats x_2, x_3 and x_4, then x_1 is beaten by x_5 and x_6. So the tournament must be the tournament T_6 of Figure 4.

We have still to prove that, if n is even and is greater than or equal to 8, then there exists a strongly connected tournament T such that there exist a Slater order O_S and a Copeland order O_C with $conc(O_S, O_C) = n + 1$. To do this, we set $n = 2p$ where p is an integer greater than or equal to 4.

Let us define the tournament T on the vertices 1, 2, …, $2p$ as follows. We build a circular tournament called T_{circ} on the set $\{4, 5, \ldots, 2p\}$ of vertices; T_{circ} is a circular tournament defined on an odd number of vertices, so T_{circ} is regular, and all the vertices have a score equal to $p - 2$. We add vertex 3 which beats the vertices 4, 5, 7, 9, ..., $2p - 1$, i.e. the vertex 4 and the vertices $2i + 1$ for $i \in \{2, 3, \ldots, p - 1\}$; vertex 3 is beaten by the other vertices between 6 and $2p$, i.e. 6, 8, ..., $2p - 2$, $2p$. We add vertex 2 which beats the vertices 3, 8, 10, 12, …, $2p$, i.e. 3 and the vertices $2i$ for $i \in \{4, 5, \ldots, p\}$; vertex 2 is beaten by the other vertices between 4 and $2p$, i.e. 4, 5, 6, 7, ..., $2p - 3$, $2p - 1$. Last, we add vertex 1 which beats vertex 6 and which is beaten by all the other vertices. Figure 5 specifies the obtained tournament for $n = 8$. In this tournament, we have that:

- the score of vertex 1 is 1;
- the score of vertex 2 is $p - 1$;
- the scores of the other vertices are p.

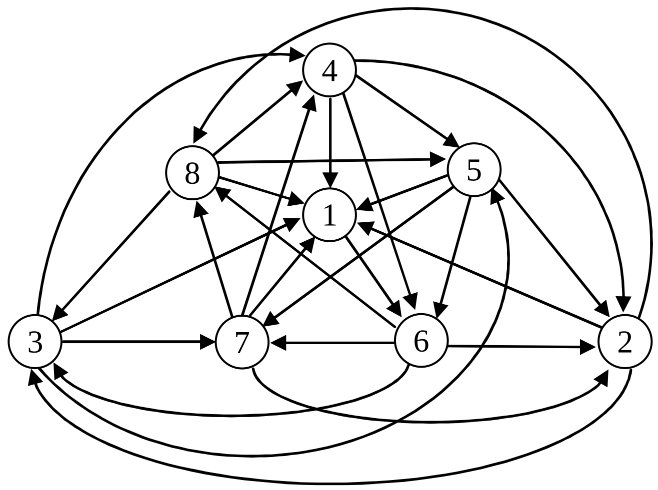

FIGURE 5. The tournament obtained for n = 8 and reaching the bound of Theorem 9.

Now, consider the Copeland order O_C = (5, 4, 3, 2p, 2p – 1, 2p – 2, …, 8, 7, 6, 2, 1) and the order O_S = (6, 7, 2, 8, 9, 10, …, 2p – 2, 2p – 1, 2p, 3, 4, 5, 1). To compute $conc(O_S, O_C)$, notice that O_S and O_C are almost reversed orders: what prevents them from being reversed orders comes from the positions of 1, 2, 6 and 7. So the concordance between these two orders involves these vertices. More precisely:

- vertex 1 brings 2p – 1 to the concordance;
- in addition to the contribution of 1, vertex 2 brings an extra concordance of 2 because of vertices 6 and 7;
- the other vertices, including 6 and 7, do not bring extra concordance.

Thus we have $conc(O_S, O_C)$ = 2p + 1 = n + 1.

To conclude, we must prove that O_S is a Slater order of T. To do this, let us compute the distance between T and O_S:

- the restriction of O_S to the vertices 4, 5, …, 2p is a Slater order of the tournament T_{circ} (see Theorem 2) of which the order is 2p – 3; this brings a contribution to the distance equal to $\frac{(2p-4)(2p-2)}{8} = \frac{(p-2)(p-1)}{2}$;
- vertex 1 brings an extra contribution to the distance equal to 1 because of vertex 6;
- vertex 2 brings an extra contribution equal to p – 2 because of the p – 2 vertices of T_{circ} other than 6 and 7 which beat it;
- vertex 3 brings an extra contribution equal to p – 3 because of the p – 3 vertices of T_{circ} other than vertices 4 and 5 and beaten by vertex 3.

The distance between T and O_S is thus equal to:

$$\frac{(p-2)(p-1)}{2} + 1 + p - 2 + p - 3 = \frac{(p-2)(p+3)}{2}.$$

To compute the Slater index of T, let us begin by enumerating the arc-disjoint circuits, according to three cases: n = 8, n = 10 and $n \geq 12$.

Assume that we have $n \geq 12$. Thanks to Theorem 8, we know that the circular tournament T_{circ}, of which the order is at least 9, owns $\frac{(p-2)(p-1)}{2}$ – 1 arc-disjoint circuits which do not use the arcs $(i, i + 1)$ for i between 3 et $2p – 1$. Besides this, T owns:

- for $i \in \{4, 5, ..., p – 1\}$, the 3-circuits $(2i, 2i + 1, 2)$, which provides $(p – 4)$ 3-circuits;
- for $i \in \{3, 4, \ldots, p\}$, the 3-circuits $(2i – 1, 2i, 3)$, which provides $(p – 2)$ 3-circuits;
- the 3-circuit $(2p, 4, 2)$;
- the 3-circuit (2, 1, 6);
- the 4- circuit (2, 3, 4, 5).

All these circuits are arc-disjoint, which gives:

$$\frac{(p-2)(p-1)}{2} - 1 + (p-4) + (p-2) + 3 = \frac{(p-2)(p+3)}{2}$$

arc-disjoint circuits.

If $n = 8$, we may point the following seven arc-disjoint circuits in T out: (2, 3, 4), (1, 6, 7), (7, 8, 3), (2, 8, 5), (3, 5, 6), (4, 6, 8), (4, 5, 7). This also provides $\frac{(p-2)(p+3)}{2}$ arc-disjoint circuits since p is here equal to 4.

If $n = 10$, we may point the following twelve arc-disjoint circuits in T out: (4, 5, 8), (4, 7, 10), (4, 6, 9), (6, 8, 10), (5, 7, 9), (1, 6, 7), (2, 3, 4), (3, 7, 8), (3, 9, 10), (3, 5, 6), (2, 8, 9), (2, 10, 5). Once again, this provides $\frac{(p-2)(p+3)}{2}$ arc-disjoint circuits since now p is equal to 5.

For any linear order O, there is at least one arc from each of these circuits which has not the same orientation in T than in O. The Slater index of T is thus at least equal to $\frac{(p-2)(p+3)}{2}$, which is also the distance between T and the previous linear order O_S. This shows that O_S is a Slater order of T, and this completes the proof of Theorem 9. ♦

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