

Links between Discriminating and Identifying Codes in the Binary Hamming Space

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Abstract

Let F^n be the binary n -cube, or binary Hamming space of dimension n , endowed with the Hamming distance, and $\mathcal{E}^n$ (respectively, $\mathcal{O}^n$) the set of vectors with even (respectively, odd) weight. For $r \geq 1$ and $x \in F^n$, we denote by $B_r(x)$ the ball of radius r and centre x . A code $C \subseteq F^n$ is said to be r -identifying if the sets $B_r(x) \cap C$, $x \in F^n$, are all nonempty and distinct. A code $C \subseteq \mathcal{E}^n$ is said to be r -discriminating if the sets $B_r(x) \cap C$, $x \in \mathcal{O}^n$, are all nonempty and distinct. We show that the two definitions, which were given for general graphs, are equivalent in the case of the Hamming space, in the following sense: for any odd r , there is a bijection between the set of r -identifying codes in F^n and the set of r -discriminating codes in F^{n+1} .

Key Words: Graph Theory, Coding Theory, Discriminating Codes, Identifying Codes, Hamming Space, Hypercube

1 Introduction

We define identifying and discriminating codes in a connected, undirected graph $G = (V, E)$, in which a *code* is simply a nonempty subset of vertices. These definitions can help, in various meanings, to unambiguously determine a vertex. The motivations may come from processor networks where we wish to locate a faulty vertex under certain conditions, or from the need to identify an individual, given its set of attributes.

In G we define the usual distance $d(v_1, v_2)$ between two vertices $v_1, v_2 \in V$ as the smallest possible number of edges in any path between them. For an integer $r \geq 0$ and a vertex $v \in V$, we define $B_r(v)$ the *ball* of radius r centred at v , as the set of vertices within distance r from v . Whenever two vertices v_1 and v_2 are such that $v_1 \in B_r(v_2)$ (or, equivalently, $v_2 \in B_r(v_1)$), we say that they *r-cover* each other. A set $X \subseteq V$ *r-covers* a set $Y \subseteq V$ if every vertex in Y is *r-covered* by at least one vertex in X .

The elements of a code $C \subseteq V$ are called *codewords*. For each vertex $v \in V$, we denote by

$$K_{C,r}(v) = C \cap B_r(v)$$

the set of codewords *r-covering* v . Two vertices v_1 and v_2 with $K_{C,r}(v_1) \neq K_{C,r}(v_2)$ are said to be *r-separated* by code C , and any codeword belonging to exactly one of the two sets $B_r(v_1)$ and $B_r(v_2)$ is said to *r-separate* v_1 and v_2 .

A code $C \subseteq V$ is called *r-identifying* [10] if all the sets $K_{C,r}(v)$, $v \in V$, are nonempty and distinct. In other words, every vertex is *r-covered* by at least one codeword, and every pair of vertices is *r-separated* by at least one codeword. Such codes are also sometimes called *differentiating dominating sets* [8].

We now suppose that G is bipartite: $G = (V = I \cup A, E)$, with no edges inside I nor A — here, A stands for *attributes* and I for *individuals*. A code $C \subseteq A$ is said to be *r-discriminating* [4] if all the sets $K_{C,r}(i)$, $i \in I$, are nonempty and distinct. From the definition we see that we can consider only odd values of r .

In the following, we drop the general case and turn to the binary Hamming space of dimension n , also called the binary n -cube, which is a regular bipartite graph. First we need to give some specific definitions and notation.

We consider the n -cube as the set of binary row-vectors of length n , and as so, we denote it by $G = (F^n, E)$ with $F = \{0, 1\}$ and $E = \{\{x, y\} : d(x, y) = 1\}$, the usual graph distance $d(x, y)$ between two vectors x and y

being called here the *Hamming distance* — it simply consists of the number of coordinates where x and y differ. The *Hamming weight* of a vector x is its distance to the all-zero vector, i.e., the number of its nonzero coordinates. A vector is said to be *even* (respectively, *odd*) if its weight is even (respectively, odd), and we denote by $\mathcal{E}^n$ (respectively, $\mathcal{O}^n$) the set of the 2^{n-1} even (respectively, odd) vectors in F^n . Without loss of generality, for the definition of an r -discriminating code, we choose the set A to be $\mathcal{E}^n$, and the set I to be $\mathcal{O}^n$. Additions are carried coordinatewise and modulo two.

Given a vector $x \in F^n$, we denote by $\pi(x)$ its parity-check bit: $\pi(x) = 0$ if x is even, $\pi(x) = 1$ if x is odd. Therefore, if $|$ stands for concatenation of vectors, $x|\pi(x)$ is an even vector. Finally, we denote by $M_r(n)$ (respectively, $D_r(n)$) the smallest possible cardinality of an r -identifying (respectively, r -discriminating) code in F^n .

In Section 2, we show that in the particular case of Hamming space, the two notions of r -identifying and r -discriminating codes actually coincide for all odd values of r and all $n \geq 2$, in the sense that there is a bijection between the set of r -identifying codes in F^n and the set of r -discriminating codes in F^{n+1} .

2 Identifying is discriminating

As we now show with the following two theorems, for any odd $r \geq 1$, any r -identifying code in F^n can be extended into an r -discriminating code in F^{n+1} , and any r -discriminating code in F^n can be shortened into an r -identifying code in F^{n-1} . First, observe that r -identifying codes exist in F^n if and only if $r < n$.

Theorem 1 *Let $n \geq 2, p \geq 0$ be such that $2p + 1 < n$, let $C \subseteq F^n$ be a $(2p + 1)$ -identifying code and let*

$$C' = \{c|\pi(c) : c \in C\}.$$

Then C' is $(2p + 1)$ -discriminating in F^{n+1} . Therefore,

$$D_{2p+1}(n+1) \leq M_{2p+1}(n). \quad (1)$$

Proof. Let $r = 2p + 1$. By construction, C' contains only even vectors. We shall prove that (a) any odd vector $x \in \mathcal{O}^{n+1}$ is r -covered by at least one codeword of C' ; (b) given any two distinct odd vectors $x, y \in \mathcal{O}^{n+1}$, there is at least one codeword in C' which r -separates them.

(a) We write $x = x_1|x_2$ with $x_1 \in F^n$ and $x_2 \in F$. Because C is r -identifying in F^n , there is a codeword $c \in C$ with $d(x_1, c) \leq r$. Let $c' = c|\pi(c)$.

If $d(x_1, c) \leq r - 1$, then whatever the values of x_2 and $\pi(c)$ are, we have $d(x, c') \leq r$; we assume therefore that $d(x_1, c) = r = 2p + 1$, which

implies that x_1 and c have different parities. Since $x_1|x_2$ and $c|\pi(c)$ also have different parities, we have $x_2 = \pi(c)$ and $d(x, c') = r$. So the codeword $c' \in C'$ r -covers x .

(b) We write $x = x_1|x_2$, $y = y_1|y_2$, with $x_1, y_1 \in F^n$, $x_2, y_2 \in F$. Since C is r -identifying in F^n , there is a codeword $c \in C$ which is, say, within distance r from x_1 and not from y_1 : $d(x_1, c) \leq r$, $d(y_1, c) > r$. Let $c' = c|\pi(c)$.

For the same reasons as above, x is within distance r from c' , whereas obviously, $d(y, c') \geq d(y_1, c) > r$. So $c' \in C'$ r -separates x and y .

Inequality (1) follows. $\square$

Theorem 2 *Let $n \geq 3, p \geq 0$ be such that $2p + 2 < n$, let $C \subseteq \mathcal{E}^n$ be a $(2p + 1)$ -discriminating code and let $C' \subseteq F^{n-1}$ be any code obtained by the deletion of one coordinate in C . Then C' is $(2p + 1)$ -identifying in F^{n-1} . Therefore,*

$$M_{2p+1}(n-1) \leq D_{2p+1}(n). \quad (2)$$

Proof. Let $r = 2p + 1$. Let $C \subseteq \mathcal{E}^n$ be an r -discriminating code and $C' \subseteq F^{n-1}$ be the code obtained by deleting, say, the last coordinate in C . We shall prove that (a) any vector $x \in F^{n-1}$ is r -covered by at least one codeword of C' ; (b) given any two distinct vectors $x, y \in F^{n-1}$, there is at least one codeword in C' which r -separates them.

(a) The vector $x|(\pi(x) + 1) \in F^n$ is odd. As such, it is r -covered by a codeword $c = c'|u \in C \subseteq \mathcal{E}^n$: $c' \in C'$, $u = \pi(c')$, and $d(x|(\pi(x) + 1), c) \leq r$. This proves that x is within distance r from a codeword of C' .

(b) Both $x|(\pi(x) + 1)$ and $y|(\pi(y) + 1)$ are odd vectors in F^n , and there is a codeword $c = c'|u \in C \subseteq \mathcal{E}^n$, with $c' \in C'$, $u = \pi(c')$, which r -separates them: without loss of generality, $d(x|(\pi(x) + 1), c) \leq r$ whereas $d(y|(\pi(y) + 1), c)$, which is an odd integer, is at least $r + 2$. Then obviously, $d(x, c') \leq r$ and $d(y, c') \geq r + 1$, i.e., there is a codeword in C' which r -separates x and y .

Inequality (2) follows. $\square$

Corollary 3 *For all $n \geq 2$ and $p \geq 0$ such that $2p + 1 < n$, we have:*

$$D_{2p+1}(n+1) = M_{2p+1}(n).$$

$\square$

3 Conclusion

We have shown the equivalence between discriminating and identifying codes; the latter being already well studied, this entails a few consequences on discriminating codes.

For example, the complexity of problems on discriminating codes is the same as that for identifying codes; in particular, it is known [9] that deciding whether a given code $C \subseteq F^n$ is r -identifying is co-NP-complete.

For yet another issue, constructions, we refer to, e.g., [1]–[3], [6], [9], [10] or [11]; visit also [12]. In the recent [7], tables for exact values or bounds on $M_1(n)$, $2 \leq n \leq 19$, and $M_2(n)$, $3 \leq n \leq 21$, are given.

Discriminating codes have not been thoroughly studied so far; let us simply mention [4] for a general introduction and [5] in the case of planar graphs.

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