On Digraph-Different Permutations

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Abstract

We extend several results on graph–different permutations of the third author and C. Malvenuto to the case of directed graphs and introduce several new open problems. This problem area is the natural extension of Sperner capacity of directed graphs to the case of infinite graphs. Sperner capacity is the key tool in determining the zero–error capacity of compound channels in case of uninformed encoder and decoder.

1. INTRODUCTION

In [3] zero–error capacity of a discrete memoryless stationary channel was generalized by restricting the input sequences to "mimick" a fixed distribution. The resulting concept of *zero–error capacity within a given type* allows to apply the method of types [4] to several important problems in extremal combinatorics and the theory of combinatorial search. All such problems can be regarded as part of zero–error information theory. In [9] and the follow–up paper [5] Shannon's graph capacity problem [12] was generalized to directed graphs. The resulting concept of Sperner capacity and the corresponding capacity within a given type gave the key to solve an intriguing problem of Rényi in combinatorial search in [6]. This direction of research started in [2] where we introduced the problem of zero–error capacity of the compound channel with uninformed encoder (but not decoder) as a unifying model for several problems in extremal combinatorics. Actually, in [6] a formally information–theoretic analogue of this problem was solved for families of directed graphs. Much to our surprise this generalization gave Nayak and Rose [11] the correct mathematical formulation and the technique of solution for the important problem of the zero–error capacity of the compound channel in case of uninformed encoder and decoder.

In this paper we introduce and study analogous problems for infinite alphabet channels. Formally, our problems are concerned with Sperner–type capacity of infinite directed graphs. To the analogy of the case of capacity within a given type we restrict the input sequences to be permutations. In case of undirected graphs these problems have been studied in the papers [7], [8] and [10]. In particular, it was shown in [10] that permutation capacities generalize the concept of graph capacity in the Shannon sense.

2. DIGRAPH–DIFFERENT PERMUTATIONS

Let N denote the set of natural numbers and let G be an arbitrary directed graph with vertex set N. We will say that two permutations, π and ρ of the first n natural numbers are G–different if there is an $i \in [n]$ such that the ordered couple of its images under these two permutations satisfies $(\pi(i), \rho(i)) \in E(G)$. We write $N(G, n)$ for the largest cardinality of a set of pairwise G–different permutations of $[n]$. We denote by $R(G)$ the (always existing but not necessarily finite) limit

$$R(G) = \lim_{n\to\infty} \frac{1}{n} \log N(G, n)$$

and call it the *permutation capacity* of G. Clearly, $R(G) \le \log \chi(G)$ where $\chi(G)$ is the chromatic number of the undirected graph underlying G. In this abstract we will concentrate on the various digraphs whose underlying undirected graph is the infinite path L. In this graph every pair of consecutive integers is adjacent. By the foregoing, all the directed graphs defined on L have capacity at most 1. Clearly, the maximum of these capacities is achieved by the *symmetric path* L_{sym} whose edge set contains two oppositely oriented edges for every edge of L. It was shown in [8] that the permutation capacity of this symmetric graph is at least $\log \sqrt[4]{10}$. No non–trivial upper bound for $R(L_{sym})$ is known. New interesting questions arise if we concentrate on *oriented paths* in which every couple of vertices gives rise to at most one oriented edge. We are especially interested in two simple oriented paths defined for L. We will denote by L_c the digraph in which every edge of L is oriented from its smaller vertex to the larger one and call it the *thrupath*. Similarly, we will denote by L_a the *alternating path* in which every edge is oriented from its odd vertex towards its even vertex. We have

Theorem 1

$$\log \frac{1+\sqrt{5}}{2} \leq \min\{R(L_a), R(L_c)\}$$

In fact, we can prove that $N(L_a, n)$ grows with n as a standard Fibonacci sequence and that $N(L_c, N)$ exhibits the same growth rate in an asymptotic sense.

It is particularly interesting to consider the generalization of the previous problems to a family of graphs in the sense of [2]. Given a finite or infinite family $\mathcal{G}$ of directed graphs with vertex set N we denote by $N(\mathcal{G}, n)$ the largest cardinality of a set of permutations of $[n]$ such that any pair of its different elements is G–different for all the graphs $G \in \mathcal{G}$. By analogy with the foregoing, we write

$$R(\mathcal{G}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{G}, n)$$

Let us now consider the family $\mathcal{L}$ of all the oriented graphs whose underlying undirected graph is the infinite path L. It is easy to see that $N(\mathcal{L}, n)$ is the largest cardinality of a set of permutations of $[n]$ with the property that for any two of its different elements some edge of L appears in both of its possible positions. This immediately yields

$$2^{\lfloor \frac{n}{2} \rfloor} \leq N(\mathcal{L}, n)$$

We believe this lower bound to be tight, with the implication that $R(\mathcal{L}) = \frac{1}{2}$.

One wonders which orientation of L (if any) forces $R(\mathcal{L})$ to be this low. More precisely, let L_{min} be the oriented graph in $\mathcal{L}$ having smallest capacity. Can we characterize these graphs? Is it true that $R(\mathcal{L}) = R(L_{min})$? We believe the answer to be negative. This would then have the further implication that the bottleneck theorem of [5] cannot be extended to permutation capacities of graph families.

We are equally interested in finding both the best and the worst orientation yielding the extremal values of these permutation capacities. We consider analogous problems for other digraphs on N including infinite tournaments. In case the corresponding capacity is infinite, we ask more refined questions about the rate of asymptotic growth of $N(G, n)$.

3. DISCUSSION

From a practical point of view, let alone for security reasons, it seems interesting to use codewords without repetition of symbols. This requires codewords on an infinite alphabet. It is clear that in such a framework the concept of *type* has to be adapted. In case of words from a finite alphabet types are probability distributions allowing to relate sequences of different length as one lets codeword lengths tend to infinity. In case of infinite alphabets we have formally simpler ways to define capacity within a given type as shown here on the example of infinite permutations. A general definition of types incorporating the case of infinite alphabets is yet to be found.

One might consider the above problems of merely mathematical interest. We are deeply convinced however, that understanding structures of difference in strings in these and similar contexts is an essential part of information theory. Experience has taught us that this kind of results quickly find their ways to "mainstream" information theory as in the case of [11].

We intend to return to the mathematical aspects of the above problems in subsequent work [1].

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