Fast InSAR Multichannel Phase Unwrapping for DEM Generation

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***Abstract*—In this paper, a method to solve the multichannel phase unwrapping problem is presented. MAP approach together with Markov Random Fields have proved to be effective, allowing to restore the uniqueness of the solution without introducing external constraints to regularize the problem. The idea is to develop a fast algorithm to unwrap the interferometric phase in the multichannel configuration, which is, in the main time, able to provide the global optimum solution. To reach this target, an *a priori* model based on Total Variation is used together with optimization algorithm based on graph-cut technique. The proposed approach has been tested both on simulated and real data. The obtained results show the effectiveness of our approach.**

I. INTRODUCTION

Interferometric Synthetic Aperture Radar (*InSAR*) systems allow to generate Digital Elevation Model (*DEM*) of Earth Surface. This is possible since there is a known relation between the interferometric SAR phase and the height of the ground [1].

When dealing with InSAR DEM reconstruction, the main problem we encounter is the phase unwrapping operation, since the measured interferometric phase is known in the principal interval (-π, π] (wrapped phase). In order to restore the relation between interferometric phase and ground height, necessary to generate the DEM, we need to unwrap the phase (i.e. to know the phase in its absolute values).

The unwrapping operation is not an easy task. In particular, if we are not in the so called *Itoh* condition [2] (the absolute value of phase difference between neighboring pixels is less than π), the phase unwrapping operation becomes an ill-posed problem. The *Itoh* condition is easily violated in real InSAR data, due both to the presence of low coherence areas and to the presence of high discontinuities. This is particularly evident in urban scenarios where buildings can be characterized by high phase jumps. To solve the phase unwrapping problem in *non-Itoh* condition, an efficient and robust method is the multichannel phase unwrapping [3]. Multichannel InSAR techniques exploit the availability of different and independent interferograms referred to the same scene, obtained using different channels (baselines or frequencies). In order to combine these different available channels, a statistical approach based on Maximum a Posteriori (*MAP*) estimation can be used. In this MAP approach, the data term is provided by the multichannel likelihood function, while the prior term is modeled by a Markov Random Field. The MAP multichannel phase unwrapping problem can be equivalently seen as an energy minimization problem.

The idea of this paper is to provide a fast and efficient (in term of global optimization) algorithm to unwrap the interferometric phase in the multichannel configuration. To reduce the computational time needed to unwrap the multichannel interferometric phase, we worked in two directions. First, we considered a non-local prior energy function and secondly, we used optimization algorithms based on *graph-cut* technique. For the *a priori* energy function, the Total Variation (*TV*) model [4] has been chosen. Even if it is a non-local model, *TV* model is well adapted when dealing with strong discontinuities. Therefore, it can be used in case of InSAR applications and it particularly well fits to urban scenario. For the optimization step, we used a graph-cut based optimization algorithm which is able to reach the global optimum (exact optimization algorithm). The approach proposed by Ishikawa [5] is used for this purpose.

Our algorithm has been tested on simulated data and on real urban data. The obtained results, for both cases, prove the effectiveness of the proposed method, both in term of short computational time, effectiveness and optimum quality (the global optimum is reached) and assess the overall interest of the proposed algorithm.

II. MAP MULTICHANNEL PHASE UNWRAPPING

In InSAR multichannel systems, a collection of interferograms is acquired using a multifrequency or a multibaseline configuration. The interferometric phase signals can be modeled as:

$$\phi_{p,c} = < \alpha_c h_p + w_{p,c} >_{2\pi} \tag{1}$$

with $p \in \{0, ..., N-1\}$, $c \in \{0, ..., M-1\}$ and

- $\alpha_c = \frac{4\pi B_c^{\perp}}{\lambda R_0 sin(\theta)}$ in case of multibaseline method,
- $\alpha_c = \frac{4\pi B^{\perp}}{\lambda_c R_0 sin(\theta)}$ in case of multifrequency method.

The index p refers to the pixel position inside the image of size N, index c to the considered channel which is one of the M possible interferograms (frequencies or baselines), w is the phase decorrelation noise, $< . >_{2\pi}$ represents the $modulo - 2\pi$ operation, λ is the wavelength and $B^{\perp}$ is the orthogonal baseline of the c^{th} SAR interferogram, θ is the SAR view angle and R_0 is the distance of the first antenna to the center of the scene. Defining these notations, the height reconstruction problem consists in estimating the height values h_p of the whole scene, using the $N \times M$ measured available wrapped phases $\phi_{p,c}$.

In the case of M statistically independent channels, the multichannel likelihood function is given by [6]:

$$F(\Phi|\mathbf{h}) = \prod_{p=0}^{N-1}\prod_{c=0}^{M-1} f(\phi_{p,c}|h_p;\alpha_c,\gamma_{p,c}) \tag{2}$$

where:

$$f(\phi_{p,c}|h_p;\alpha_c,\gamma_{p,c}) = \frac{1}{2\pi}\frac{1-|\gamma_{p,c}|}{1-d^2}\Big(1+\frac{dcos^{-1}(-d)}{(1-d^2)^{1/2}}\Big)$$
$$d = |\gamma_{p,c}|cos(\phi_{p,c} - \alpha_c h_p)$$

is the singlechannel likelihood function and $\gamma_{p,c}$ is the coherence coefficient that depends on pixel p and on channel c. In (2) $\mathbf{\Phi} = [\mathbf{\Phi}_0^T\mathbf{\Phi}_1^T...\mathbf{\Phi}_{N-1}^T]^T$ is the vector collecting all available wrapped phase values, $\mathbf{\Phi}_p = [\phi_{p,0}\phi_{p,1}...\phi_{p,M-1}]^T$ is the vector of the wrapped phases measured in pixels p for the M different channels and $\mathbf{h} = [h_0h_1...h_{N-1}]^T$ is the vector containing the ground elevation values. Following the Bayes law, the Multichannel *MAP* estimation solution is given by:

$$\hat{\mathbf{h}} = \text{argmax}_{\mathbf{h}} F(\Phi|\mathbf{h})g_\beta(\mathbf{h}) \tag{3}$$

where the function $g(.)$ is the prior probability of $\mathbf{h}$. A Markov Random Field is used to model $\mathbf{h}$, whose expression is given in this case by:

$$g_\beta(\mathbf{h}) = \frac{1}{Z(\beta)}e^{-E_{prior_\beta}(\mathbf{h})} \tag{4}$$

where $Z(\beta)$ is a normalization factor called the partition function, $E_{prior}(.)$ is the so-called energy function expressing relationship between pixels and β is the hyperparameter [7] which is used to tune the prior model. When β is a scalar, the model is a non-local model, otherwise it is defined as a local model. The prior energy function is defined in such a way to impose some constraint on the neighboring pixels. Different prior energy functions can be used to model our problem. For our purpose, we chose the *TV* model, which is a non-local model that presents some interesting features. In the next section the *TV* model is analyzed.

III. Energy function model

In order to develop a fast and robust multichannel phase unwrapping algorithm the *TV* model has been used for the prior energy. This prior model, introduced in 1992 [4], is one of the most used prior model in image processing due to its adequation to different contextual information. In SAR applications, *TV* model is mainly used for image restoration [8]. Our proposal is to apply it to the phase unwrapping problem.

The prior energy corresponding to the discretization of *TV* can be written [9] as follows:

$$E_{prior} = \beta\sum_{p\sim q} w_{p,q}|h_p - h_q| \tag{5}$$

where $p \sim q$ denotes two neighboring pixels p and q. The coefficient $w_{p,q}$ depends on neighbor connexity (1 for $4 - connexity$ and $\frac{1}{\sqrt{2}}$ for the 4 diagonal ones in the case of $8 - connexity$). Note that in this expression, β is a scalar (a single hyperparameter is used for the whole image). This makes the *TV* model a non-local model. The choice of a non-local prior energy is done in order to make the algorithm faster since it avoids the estimation of a vector of local hyperparameters as in [3].

Between the existing non-local prior energy models, *TV* has been chosen because its main advantage, towards other non-local prior energy models, is that it does not penalize discontinuities in the image while simultaneously not penalizing smooth functions either [4]. *TV*, in fact, looks for an approximation of the original noisy image which has minimal total variation but without particular bias to discontinuity or smooth solution. As the *TV* prior model is well adapted when dealing with strong discontinuities, it can be used in case of InSAR applications and particularly it well fits to urban areas. So, in order to develop a fast algorithm, we considered this *a priori* energy. We note that this choice is not as good and powerful as the local one proposed in [3], since it is not local. However, compared to other non-local models, it is the best for our purpose.

Given the *TV* prior energy model (5), the function to be minimized to recover the multichannel height reconstruction solution is given by:

$$\begin{aligned}\hat{\mathbf{h}} &= \text{argmax}_{\mathbf{h}} F(\Phi|\mathbf{h})g_\beta(\mathbf{h})\\ &= \text{argmin}_{\mathbf{h}} \sum_p\sum_c -\log f(\phi_{p,c}|h_p;\alpha_c,\gamma_{p,c}) +\\ &+ \beta\sum_{p\sim q} w_{p,q}|h_p - h_q| \end{aligned}\tag{6}$$

In the next section the proposed method to minimize this energy in order to obtain the global optimum is explained.

IV. Fast and exact optimization algorithm

Fast and exact optimization algorithms are needed in many computer vision problems. In recent years, new energy optimization approaches have been proposed [5], [9], [10] based on graph-cut technique, which has become very popular due to its low time computation. Compared to the classical optimization algorithm Simulated Annealing [11], it provides comparable results with much less computational time and compared to the deterministic algorithm ICM [12], it avoids the risk of being trapped in a local minimum of the energy that is far from the global one. We use, in this work, the graph-cut optimization algorithm proposed by Ishikawa [5], that is able, under certain conditions, to provide the global optimum of the considered problem.

Suppose $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed graph with non negative edge weights, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ the set of edges. This graph has two special vertices (terminals) called the *source* s and the *sink* t. We define an s-t-cut $\mathcal{C} = \{\mathcal{S}, \mathcal{T}\}$ as a partition of the vertices into two disjoint sets $\mathcal{S}$ and $\mathcal{T}$ such that $s \in \mathcal{S}$ and $t \in \mathcal{T}$. The cost of this cut is the sum of weights (costs) of all edges that go from $\mathcal{S}$ to $\mathcal{T}$.

$$|\mathcal{C}_{s,t}| = \sum_{\substack{u\in\mathcal{S}\\ v\in\mathcal{T}\\ (u,v)\in\mathcal{U}}} w(u,v) \tag{7}$$

The minimum s-t-cut problem consists in finding a cut $\mathcal{C}$ with the smallest cost. This problem is exactly equivalent to its dual problem which consists in computing the maximum flow growing from the source to the sink [13]. Many algorithms have been proposed to solve the maximum flow problem with a polynomial time [14]. In our work, we will use the maximum flow algorithm given by [15] which is the most adapted to computer vision problems.

The exact optimization algorithm proposed by Ishikawa is based on two hypothesis: convexity of the *a priori* energy function and a linear order on the label set. We suppose in the following that the labels can be represented as integers in the range $\{0, 1, .., K-1\}$, where K is the size of the label set. Ishikawa method is based on computing a minimum cut in a particular graph. This graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ contains $N \times K$ nodes (N is the size of the image and K is the size of the label set) denoted by $\Big\{v_{p,i}; p = 0..N-1; i = 0..K-1\Big\}$ plus two special nodes s and t. For each pixel p, we associate K nodes $\Big\{v_{p,i}; i = 0..K-1\Big\}$ that represent all the possible labels that the pixel p can take. Besides, $\mathcal{G}$ contains three families of edges $\mathcal{E} = \mathcal{E}_\mathcal{D} \cup \mathcal{E}_\mathcal{C} \cup \mathcal{E}_\mathcal{I}$.

$\mathcal{E}_\mathcal{D}$ is a set of directed edges called data edges. It represents the multichannel likelihood energy terms and is defined by:

$$\begin{aligned} \mathcal{E}_\mathcal{D} &= \bigcup_{p=0..N-1} \mathcal{E}_D^p \\ \mathcal{E}_D^p &= \Big\{(s, v_{p,0})\Big\} \cup \Big\{(v_{p,i}, v_{p,i+1}); i = 0..K-2\Big\} \\ &\quad \cup \Big\{(v_{p,K-1}, t)\Big\} \end{aligned}$$

Each edge in $\mathcal{E}_D^p$ has a capacity defined by:

$$c(s, v_{p,0}) = +\infty \tag{8}$$
$$c(v_{p,i}, v_{p,i+1}) = F(p,i); i = 0..K-2 \tag{9}$$
$$c(v_{p,K-1}, t) = F(p, K-1) \tag{10}$$

where $F(p,i)$ is the multichannel likelihood value that the pixel p assumes for all the possible heights $\{0, 1, ..., K-1\}$, i.e. $F(p,i) = \sum_c -\log f(\phi_{p,c}|h_p = i; \alpha_c, \gamma_{p,c})$.

$\mathcal{E}_\mathcal{C}$ is a set of directed edges called constraint edges defined by:

$$\begin{aligned} \mathcal{E}_\mathcal{C} &= \bigcup_{p=0..N-1} \mathcal{E}_C^p \\ \mathcal{E}_C^p &= \Big\{(v_{p,i+1}, v_{p,i}); i = 0..K-2\Big\} \end{aligned}$$

The capacity of each edge in $\mathcal{E}_C^p$ is set to be infinity to ensure that only one data edge is in the minimum cut for each pixel p. This constraint ensures the one-to-one correspondence between configurations of the MRF and cuts on the graph.

Finally, $\mathcal{E}_\mathcal{I}$ is a set of interaction edges between all neighbor pixels defined by:

$$\mathcal{E}_\mathcal{I} = \{(v_{p,i}, v_{q,j}); (i,j) = 0..K-1, (p,q) = 0..N-1; p \sim q\}$$

To obtain a correspondence between interactions edges in a cut and the energy regularization terms, capacity of each edge in $\mathcal{E}_\mathcal{I}$ connecting two neighboring pixels p and q is set to:

$$c(v_{p,i}, v_{q,j}) = \frac{1}{2}\Big(g(i-j+1) - 2g(i-j) + g(i-j-1)\Big) \tag{11}$$

where the function $g(.)$ is defined by:

$$E_{prior}(h_p, h_q) = w_{p,q} g(h_p - h_q) \tag{12}$$

The convexity of $g(.)$ is necessary and sufficient for the non negativeness of all edge capacities in $\mathcal{G}$ which is sufficient for computation of minimum cuts in polynomial time [5]. This hypothesis is satisfied considering $E_{prior} =$ *TV*.

This graph construction ensures that the minimum cut on the graph provides the optimal configuration for our *PU* problem. In term of memory occupation, for a general prior energy function, this graph has a number of vertices that is $O(N \times K)$ and a number of edges that is $O(N \times K^2)$. However, in the case of *TV* prior energy function, the number of edges is reduced dramatically and reaches $O(K \times N)$.

V. Experimental results

In order to prove the effectiveness of the proposed method, we present some results obtained both on simulated and real data. The results presented were obtained with MATLAB coding (max-flow algorithm is implemented in C++ [1]).

A. Hyperparameter β estimation

As we deal with regularization problem, we need to estimate the hyperparameter β in order to avoid over or under regularization. Optimal value of the hyperparameter may differ from one application to other and the range of possible values depends on both the log-likelihood and the prior terms and it is usually large. So, automatic method to estimate this parameter is necessary.

Various methods have been developed for the optimal selection of the regularization parameter, especially in the case of Tikhonov regularization form: discrepancy principle [16], generalized cross-validation, $L-Curve$ [17]. The $L-Curve$

[1] by V. Kolmogorov, http://www.cs.cornell.edu/People/vnk/software.html

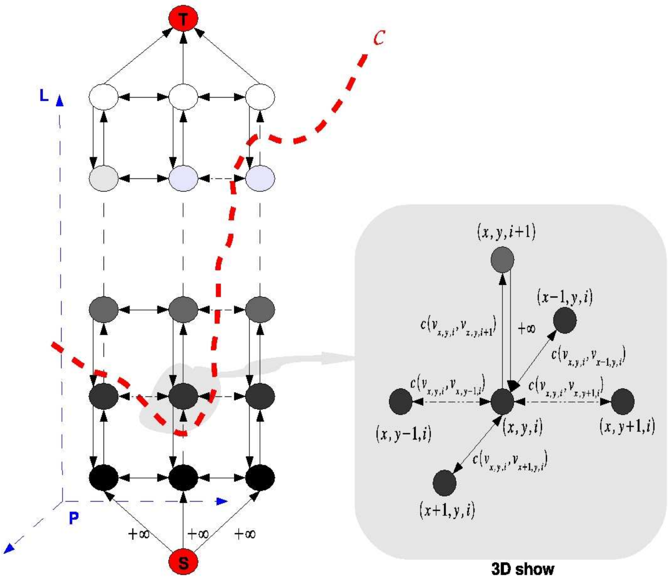

Fig. 1. Ishikawa graph for exact optimization in case of *TV* prior. On the left, a part of the graph $\mathcal{G}$ defined on three pixels. Nodes are colored related to the label set $\mathcal{L}$ in a linear order. We show also two types of edges, the dotted ones are in the cut, whereas the continuous ones are not in the cut. The red line shows the cut. A part of the graph is highlighted on the right. We distinguish the three families: data edges (vertical edges oriented to the top), constraint edges (verticals edges oriented to the bottom) and penalty edges connecting all label nodes of two 4-connexity neighbor pixels $\Big(p = (x,y)\ ;\ q \in \{(x,y+1),(x,y-1),(x-1,y),(x+1,y)\Big\}$. i refers to the level of node label.

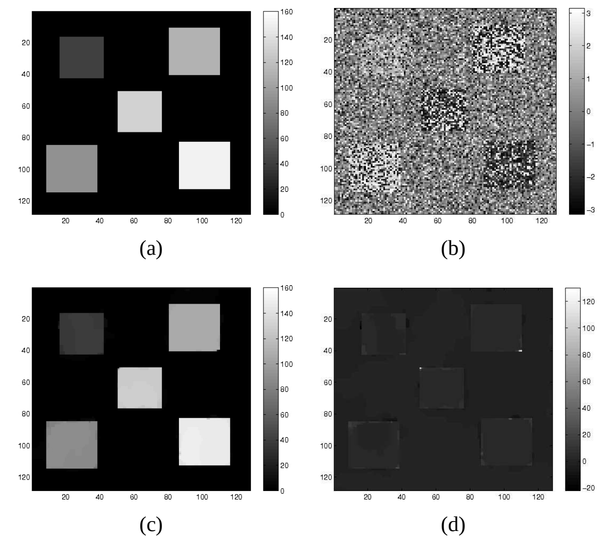

Fig. 2. *MCPU* results based on *TV* model and Ishikawa exact optimization algorithm. (a) Original urban profile, (b) Noisy interferogram at $5GHz$ with $\gamma = 0.5$, (c) Reconstructed profile, (d) Difference between ground truth and estimated profiles.

method has gained attention in recent years [18], [19]. The $L-Curve$ is the graphical representation of the regularization term with respect to the likelihood energy term. In this curve, under-regularization can be seen in the steep part of the curve, where the regularization energy term can be largely improved with minor likelihood modification. Whereas, over regularization can be seen in the slowly varying part of this curve, where no longer improvement is possible whatever the likelihood price. The corner of the $L - Curve$, which is the maximum curvature point, corresponds to a good trade-off between under and over regularization. To fix it automatically, we used the triangle method described in [20].

B. Simulated data

Based on simulated data, we tried to prove the effectiveness of the proposed approach concerning both the proposed model for *PU* problem and the graph-cut based optimization algorithm. We performed two experiments. The first one shows the effectiveness of this method when we deal with simulated urban scenario and the second one shows the power of this method when very difficult interferograms have to be unwrapped.

In the first experiment, we considered a height profile image (128×128 pixels) with a maximum height of $160m$ exhibiting discontinuous areas (Fig.2(a)) that characterizes an urban area. We used two frequencies ($5GHz$ and $9GHz$) to generate interferograms and we added interferometric noise with a coherence of $\gamma = 0.5$. In Fig.2(b), we show the $5GHz$ noisy interferogram. Moreover, for each working frequency we generated 4 azimuth looks allowing to generate a total of $M = 8$ independent interferograms. It is important to note that the profile is ambiguous for both the working frequencies. In fact, there are phase jumps of about 1.3π at $5GHz$ and 2.4π at $9GHz$ which violates the *Itoh* condition. For this reason, a classical single frequency phase unwrapping method [21] would fail. The multichannel approach can overcome this problem.

The normalized square reconstruction error is equal 2×10^{-3}, using the estimated hyperparameter $\beta = 0.7$. To obtain the result presented in figure 2(c), the algorithm took 24 seconds. For the same data set, the multichannel phase unwrapping method proposed in [3], based on a local model and on ICM algorithm for the optimization step provides similar results but in approximately 300 seconds.

We can show also the effect of *TV* prior when dealing with discontinuities. The Total Variation regularization is able to well preserve the discontinuities in original image, even if it is a local model. Anyway, a trade off of the *TV* is the loss of contrast between gray level of an homogeneous object and its background. This is a well known problem related to our energy function model which was discussed in other works solving image restoration problem such as [22].

In the second experiment, we considered a Gaussian reference profile (100×160 pixels) with maximum height of 60 rad (Fig.3(a)). We used the same system parameters as the previous experiment. The interferogram is simulated using a coherence equal to 0.6. In Fig.3(b), we show the $5GHz$ noisy interferogram. The considered interferogram is noisy, shows large phase rate and presents a large discontinuity. In these

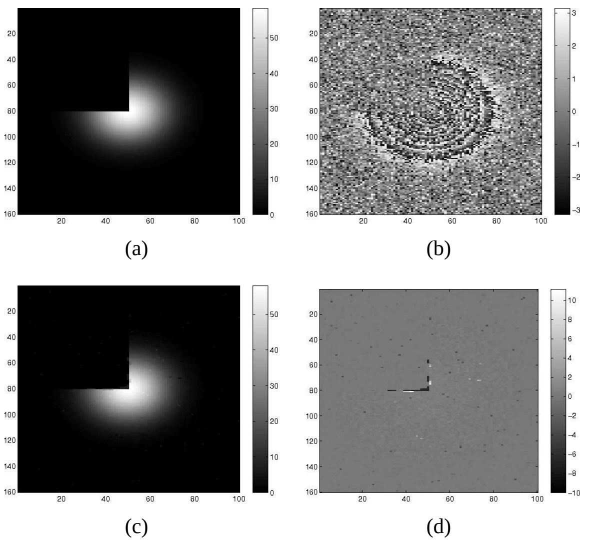

Fig. 3. *MCPU* results based on *TV* model and Ishikawa exact optimization algorithm. (a) Original Gaussian profile, (b) Noisy interferogram at $5GHz$ with $\gamma = 0.5$, (c) Reconstructed profile by our approach, (d) Difference between ground truth and estimated profiles by our method

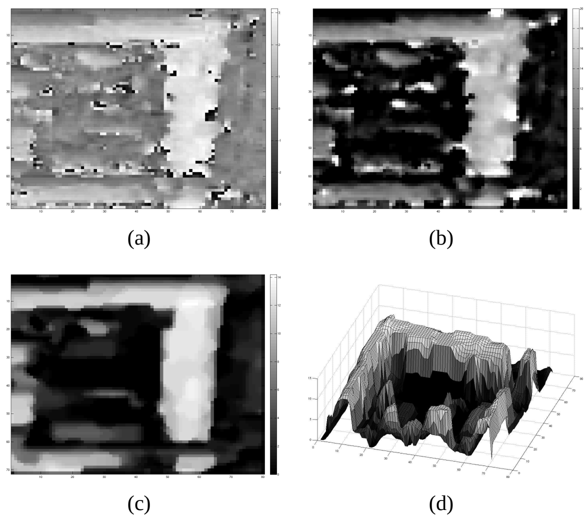

Fig. 4. 3D reconstruction of real data. (a) Noisy interferogram (smallest baseline), (b) reconstruction with branch-cut algorithm, (c) reconstruction with the proposed multichannel approach, (d) 3D view of the reconstruction with the proposed multichannel approach.

condition a classic phase unwrapping algorithm would fail to retrieve the correct solution. Even the method proposed by Bioucas [23], based on a L^p norm phase unwrapping method using graph-cuts, is not able to properly unwrap the considered interferogram. In our case, the unwrapping operation is possible since the multichannel configuration is exploited. Figures 3(c) 3(d) represent respectively the reconstruction obtained by our proposed approach and the difference between the original profile and the estimated profile. The normalized square reconstruction error is of 2.2×10^{-3}, using the estimated hyperparameter $\beta = 0.25$.

C. Real data

We tested our new proposed algorithm on real data set of an urban scenario. We used a set of 8 L-Band E-SAR interferograms (two interferograms for each of the four available polarizations) acquired on the city of Dresden. The smallest baseline is of about $10m$ and the biggest is of about $40m$. We applied the fast unwrapping approach previously exposed. Note that the height of the considered building is less than the ambiguous height in the smallest baseline interferogram. Ambiguity height is $h_{amb} = 37.5m$ in the first interferogram and $h_{amb} = 11.1m$ in the second. Anyway, it can be considered as a good test, since due to the presence of noise, the Itoh condition is violated in some areas, as it can be seen in Fig.4(a). So, applying a classic single channel approach, the solution will show some errors in that areas Fig.4(b). Using the proposed approach, it is possible to retrieve the correct height of the building. For example, note that the height of the roof building is almost constant Fig. 4(c), differently from Fig.4(b). Moreover, using these interferograms, it is possible to test the goodness of the *TV* prior model concerning the capacity of recovering discontinuities. Figure 4(c) and 4(d) show how efficiently the contours of the building are recovered.

VI. Conclusion

In this paper, we developed a new multichannel *MAP* phase unwrapping methodology based on exact graph-cut optimization algorithms (Ishikawa algorithm) and Total Variation prior model. The proposed algorithm is able to provide the solution of the multichannel phase unwrapping problem in very short time. Moreover, with Ishikawa optimization algorithm, we are sure to reach the exact energy optimum. We have tested this approach on simulated data and we obtained good results both in term of reconstruction error and computational time. Comparisons with other approaches in the literature show the contribution of this paper in the considered field, since it provides both robust model and fast running algorithm. As it proved to be effective dealing with high discontinuities, we also tested the proposed method with real *InSAR* data of urban scenarios.

VII. Acknowledgments

The authors wish to thank the German Aerospace Center (*DLR*) for supplying the E-SAR data.

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