Ann Oper Res (2010) 175: 107–158
DOI 10.1007/s10479-009-0648-7

An updated survey on the linear ordering problem for weighted or unweighted tournaments

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Published online: 19 November 2009

Abstract In this paper, we survey some results, conjectures and open problems dealing with the combinatorial and algorithmic aspects of the linear ordering problem. This problem consists in finding a linear order which is at minimum distance from a (weighted or not) tournament. We show how it can be used to model an aggregation problem consisting of going from individual preferences defined on a set of candidates to a collective ranking of these candidates.

Keywords Aggregation of preferences · Voting theory · Social choice · Linear ordering problem · Kemeny's problem · Slater's problem · Median order · Reversing set · Feedback arc set · Acyclic subgraph · Optimal triangulation · Graph theory · Tournament solutions · Complexity · Combinatorial optimization · Combinatorics

1 Introduction: from individual preferences to tournaments

Although the two problems on which this paper is based (*Kemeny's problem*, *Slater's problem*) have a same graph theoretical modelling, their origins and their meanings are different. They deal with *linear orders* (i.e., complete antisymmetric transitive binary relations) and with *tournaments* (i.e., complete antisymmetric binary relations; notice that a linear order is a transitive tournament and conversely). We already considered these problems in Charon and Hudry (2007).

The first one, Kemeny's problem (Kemeny 1959), is also attributed to Condorcet (1785) (see Young 1988 for a deep analysis of Condorcet's work and the reasons why this problem can be attributed to him; for a broader historical point of view and references on the context,

This is an updated version of the paper that appeared in *4OR*, 5(1), 5–60 (2007).

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see Barthélemy and Monjardet 1981; Black 1958; Hudry et al. 2009; McLean 1995; McLean and Hewitt 1994; McLean et al. 2008 and McLean and Urken 1995, 1997). As this problem can be stated in different ways (see Monjardet 1990 and below), several authors rediscovered this problem. Kemeny seems to be the first one among them, hence the name of Kemeny's problem, or sometimes *Kemeny's rule*. It can be stated as follows, if we adopt voting theory as the context: given a collection (called a *profile*) Π of m individual preferences O_i ($1 \leq i \leq m$) which are linear orders defined on a same finite set X of n candidates, determine a collective preference O^* which is also a linear order aggregating the preferences of Π "as accurately as possible" (of course, this notion of accuracy must be specified mathematically with respect to a criterion; this one is defined below).

The second problem, Slater's problem (Slater 1961), is the following: given a tournament T defined on the set X, determine a linear order O^* fitting T "as well as possible". For instance, T may represent the "preference" (not necessarily transitive) of a unique voter v who can be inconsistent in the sense that v may prefer a candidate x of X to another candidate y of X, and y to a third candidate z of X, and z to x. Or T may be the *majority tournament* representing the result of a paired-comparison method, as the one proposed by Condorcet at the end of the XVIIIth century (Condorcet 1785). In such a method, m voters rank the elements of X. Then the majority relation T is built on X: for two candidates x and y of X, we have xTy if x is preferred to y by a majority of voters. If there is no tie, T is a tournament which is not necessarily a linear order, even if the individual preferences of the voters are linear orders: this is the well-known "Voting Paradox", also called "effet Condorcet" in French (see Guilbaud 1952).

To define the criterion to optimize in both cases, we consider the *symmetric difference distance* δ (also linked to the Kendall distance, see Kendall 1938) which measures the number of disagreements between two relations R and S defined on a same set (see for instance Barthélemy and Monjardet 1981 for its properties):

$$\delta(R, S) = |\{(x, y) \text{ with } [xRy \text{ and not } xSy] \text{ or } [xSy \text{ and not } xRy]\}|.$$

If we consider now a profile $\Pi = (T_1, T_2, \ldots, T_m)$ (in which the relations T_i with $1 \leq i \leq m$ are linear orders for Kemeny's problem, or in which $m = 1$ and T_1 is a tournament for Slater's problem), we define the *remoteness* Δ (Barthélemy and Monjardet 1981) between Π and a relation O by:

$$\Delta(\Pi, O) = \sum_{i=1}^{m} \delta(T_i, O).$$

This remoteness Δ may be interpreted as the total number of disagreements between Π and O.

In order to unify the statements of Kemeny's problem and Slater's problem, we will assume in the sequel that all the relations T_i ($1 \leq i \leq m$) of the profile are tournaments (which are assumed to be transitive for Kemeny's problem). Let us remind that, in a tournament T, for any two candidates x and y with $x \neq y$, one and only one of the following two possibilities occurs: xTy or yTx (see Laslier 1997; Moon 1968; Reid 2004 or Reid and Beineke 1978 for basic definitions and results about tournaments; see also McKey 2006 for a catalogue of the non-isomorphic tournaments with $n \leq 10$ or of special families of tournaments). As noticed at the beginning of this section, the set of all the tournaments defined on X contains the linear orders also defined on X. So, it may happen that the considered individual preferences are linear orders as well. In both problems, we shall consider that the relation O^* that we look for is also a linear order defined on X (though other partially ordered structures may be considered; see for instance Caspard et al. 2007 for partially ordered structures). Then we obtain a *combinatorial optimization problem*: compute a linear order O^* minimizing $\Delta(\Pi, O)$ in the first case or $\delta(T, O)$ in the second case, over the set of linear orders defined on X. In the first case, O^* will be called a *median order* of Π (Barthélemy and Monjardet 1981) or a *Kemeny order* of Π if the individual preferences are assumed to be linear orders. The problem itself has different names, as *Kemeny's problem* (or *rule*), *median order problem, linear ordering* (or *order*) *problem*, *linear arrangement problem*, *permutation problem*... (for references, see for instance Barthélemy and Monjardet 1981, 1988; Grötschel et al. 1984a; Huber 1976; Jünger 1985; Lenstra 1977; Monjardet 1973, 1979; Reinelt 1985; Young 1978; see below for other equivalent statements and thus other names, and for references; see also Bertacco et al. 2008; Buchheim et al. 2009; Duarte et al. 2009 and Righini 2008 for variants of this problem). In the second case, O^* will be called a *Slater order* of T, as Slater (1961) defined this fitting problem.

Consequently, Kemeny's problem and Slater's problem may be stated as follows:

Problem 1 (*Kemeny's problem* when the profile contains linear orders, *median order problem,* or *linear ordering problem*; *Slater's problem* when the profile contains $m = 1$ tournament) Given a profile Π of m tournaments (or of m linear orders) defined on a same set X, compute a linear order O^* minimizing Δ with respect to Π on the set Ω of the linear orders defined on X:

$$\Delta(\Pi, O*) = \min_{O\in\Omega} \Delta(\Pi, O).$$

It is usual to represent a preference R defined on X by a graph (for reference on graph theory, see for instance Bang-Jensen and Gutin 2001 or Berge 1985). Its vertices are the elements of X, and there is an arc (i.e., a directed edge) (x, y) from a vertex x to a vertex y when x is preferred to y with respect to R, i.e., when we have xRy. The properties of the graph are the ones of R: it can be antisymmetric, complete, transitive, and so on. This allows us to consider tournaments and linear orders as graphs. Similarly, the profile $\Pi = (T_1, T_2, \ldots, T_m)$ can also be represented by a directed, weighted, complete, symmetric graph $G_\Pi = (X, U_X)$: its set of vertices is X, and G_Π contains all the possible arcs except the loops (i.e., $U_X = X \times X - \{(x, x) \text{ for } x \in X\}$). The weights of the arcs (x, y) give the intensity of the preferences for x over y. In order to compute it, we set (see Barthélemy and Monjardet 1981 for instance): $t^j_{xy} = 1$ if xT_jy (i.e., if x is "preferred" to y by voter j) and $t^j_{xy} = 0$ otherwise. Similarly, we set $o_{xy} = 1$ if xOy and $o_{xy} = 0$ otherwise. Then we obtain:

$$\begin{aligned}\delta(T_j, O) &= \sum_{(x,y)\in X^2} |t^j_{xy} - o_{xy}| = \sum_{(x,y)\in X^2} [t^j_{xy} - o_{xy}]^2 \\ &= \sum_{(x,y)\in X^2} \left[(t^j_{xy})^2 - 2t^j_{xy}o_{xy} + (o_{xy})^2\right] = \sum_{(x,y)\in X^2} [t^j_{xy} + (1 - 2t^j_{xy})o_{xy}]\end{aligned}$$

and

$$\Delta(\Pi, O) = C - \sum_{(x,y)\in X^2} m_{xy}o_{xy}$$

where $C = \sum_{j=1}^m \sum_{(x,y)\in X^2} t^j_{xy} = \frac{mn(n-1)}{2}$ is a constant and with $m_{xy} = 2\sum_{j=1}^m t^j_{xy} - m = \sum_{j=1}^m t^j_{xy} - \sum_{j=1}^m t^j_{yx}$. Notice that m_{xy} measures the number of voters who prefer x to y

minus the number of voters who prefer y to x. It is also equal to twice the gap between the number of voters who prefer x to y and the majority. It is a non-positive or non-negative integer with the same parity as m. Moreover we have: $m_{xy} + m_{yx} = 0$. Then we weight the arcs (x, y) of G_Π by m_{xy}.

Example Consider the following example, with $n = 3$ and $m = 4$. Let a, b, and c denote the three candidates. Assume that the preferences of the four voters are respectively given by the four linear orders $a > b > c$, $a > c > b$, $b > a > c$, $b > c > a$, with the agreement that, for x and y belonging to $\{a, b, c\}$, $x > y$ means that x is preferred to y by the considered voter. Then, the values of the quantities m_{xy} are: $m_{ab} = 0$, $m_{ba} = 0$, $m_{ac} = 2$, $m_{ca} = -2$, $m_{bc} = 2$, $m_{cb} = -2$.

Thus it is easy to associate a graph with each profile of tournaments or of linear orders. Conversely, the question arises: which weighted graphs $G = (X, U_X)$ can be associated with a profile Π of tournaments or of linear orders? Debord (1987a, 1987b) provides such characterisations, at least, for profiles of linear orders, when the number of linear orders is large enough (see also Erdös and Moser 1964; McGarvey 1953 and Stearns 1959).

Theorem 1 *The graph $G = (X, U_X)$ weighted by the (non-positive or non-negative) integers m_{xy} represents a profile Π of m tournaments if and only if the following conditions are fulfilled, where M denotes the highest weight of G ($M = \mathrm{Max}_{(x,y)\in U_X} m_{xy}$):*

1. *all the weights m_{xy} have the same parity,*
2. *m has the same parity as the weights m_{xy},*
3. *$m \geq M$,*
4. *$\forall (x, y) \in U_X, m_{xy} = -m_{yx}$.*

Theorem 2 *The graph $G = (X, U_X)$ weighted by the (non-positive or non-negative) integers m_{xy} represents a profile Π of m linear orders if the following conditions are fulfilled, where $N(G)$ is the number of linear orders in Debord's construction:*

1. *all the weights m_{xy} have the same parity,*
2. *m has the same parity as the weights m_{xy},*
3. *$m \geq N(G)$,*
4. *$\forall (x, y) \in U_X, m_{xy} = -m_{yx}$.*

Notice that, for Theorem 1, M is the lowest possible value of m. For a profile of linear orders, the number $N(G)$ of linear orders required in Debord's construction is about $\sum_{m_{xy}>0} m_{xy}$ when this quantity is not equal to 0 (otherwise, when all the quantities m_{xy} are equal to 0, the minimum number of linear orders is equal to 2; see Hudry 2008 for more details). It is sometimes possible to find a profile with fewer linear orders associated with G (but still greater than or equal to M). But minimizing the number of linear orders required for the construction of such a profile seems quite difficult in general.

Moreover, if we assume that the weights m_{xy} are upper-bounded by a polynomial in n, then the construction of Π can be done in polynomial time with respect to the size of G. As it is the case in the papers dealing with complexity issues, from the point of view of the theory of NP-completeness and because of this polynomial link, the complexity of our aggregation problems is the same as the graph theoretical problems that we are going to tackle below.[1]

The previous computations show that minimizing Δ is the same as computing the binary variables o_{xy} in order to maximize $\sum_{(x,y)\in X^2} m_{xy}o_{xy}$ while the variables o_{xy} define a linear order. Thus we obtain a 0-1 linear programming formulation of our problem (see for instance Arditti 1984; de Cani 1969; Hudry 1989; Nishihara et al. 1989; Tucker 1960; Wakabayashi 1986; Younger 1963; more generally, see Barthélemy and Monjardet 1981 and below for references; for other equivalent expressions, see Monjardet 1990):

$$\text{Maximize} \sum_{(x,y)\in X^2} m_{xy}o_{xy}$$

under the constraints:

$$\begin{cases} \forall (x,y)\in X^2,\ x\neq y, \quad o_{xy}+o_{yx}=1 \quad \text{(antisymmetry and completeness)},\\ \forall (x,y,z)\in X^3,\ x\neq y\neq z\neq x, \quad o_{xy}+o_{yz}-o_{xz}\leq 1 \quad \text{(transitivity)},\\ \forall (x,y)\in X^2, \quad o_{xy}\in\{0,1\}. \end{cases}$$

From a graph theoretic point of view, we may also see this problem as consisting in drawing from G_Π a subset of arcs with a maximum total weight and defining a linear order, or as deleting $\frac{n(n-1)}{2}$ arcs from G_Π with a minimum total weight so that the $\frac{n(n-1)}{2}$ remaining arcs of G_Π define a linear order with a maximum total weight.

In order to obtain a formulation with a tournament as the representation of the profile, it is sufficient to see that G_Π contains redundant information: $\forall (x,y)\in X^2$ with $x\neq y$, $m_{xy}=-m_{yx}$ (notice anyway that, if the relations of the profile are not assumed to be tournaments, this relation is not necessarily true and the following cannot be applied systematically). Then we may keep only one arc (x,y) or (y,x), for instance the one of which the weight (m_{xy} or m_{yx}) is non-negative. If m_{xy} and m_{yx} are equal to 0, we keep one of the arcs arbitrarily. In other words, we direct the arc from x to y when there exists a majority of voters who prefer x to y. This leads to the *majority tournament* associated with Π. Then we obtain a tournament T_Π of which the weights are non-negative, and with a same parity. As we look for a linear order as the median relation of the profile, it is equivalent to say that we remove an arc (x,y) or that we keep the arc (y,x) in G_Π. With respect to T_Π, the search of a median order is the same as finding a subset of arcs with a minimum total weight such that reversing these arcs (x,y) in T_Π makes T_Π become transitive: these arcs are the ones that we kept to build T_Π while it would have been necessary to choose (y,x) to obtain a median order of Π.

Seen from the graph theoretic point of view, Problem 1 can also be stated as follows, where $\mathbb{N}$ denotes the set of non-negative integers:

[1] It is usual to represent the profile Π as a multiset $\Pi=(T_1,T_2,\dots,T_m)$ in which two elements T_i and T_j may be equal even if i and j are distinct. But there exists another representation of the profile Π. Since the elements of Π may be the same, we may consider only the different elements $R_1,R_2,\dots,R_q$ of Π, where q denotes this number of different elements. We then gather all the voters sharing a same preference, i.e. the indices of the preferences T_j $(1\leq j\leq m)$ corresponding with a same element of Π. If m_i $(1\leq i\leq q)$ denotes the number of voters associated with a same element R_i $(1\leq i\leq q)$ of Π (notice the equality $\sum_{i=1}^q m_i=m$), we associate this number m_i of occurrences with each element R_i of Π. Then Π can be described as the set of such pairs: $\Pi=\{(R_1,m_1);\ (R_2,m_2);\dots;(R_q,m_q)\}$. Such a representation does not involve the same size for the data when m is large with respect to n. Anyway, usually, the complexity results are the same for the two representations, because their proofs stand even if m is bounded by a polynomial in n, and in this case these is no qualitative difference between the two representations. Anyway, for some variants of Kemeny's problem, these different representations may involve different locations inside the polynomial hierarchy, as shown by Hemaspaandra et al. (2005) (see also Hudry 2009c).

Problem 2 (*Minimum reversing set*) Given a tournament $T = (X, U)$ weighted by p defined from U into $\mathbb{N}$, find a subset V of U such that $\sum_{(x,y)\in V} p(x, y)$ is minimum and such that the tournament obtained by reversing in T the elements of V is transitive.

Remark 1 From what is said above, it appears that Problems 1 and 2 differ by two aspects. The first one, already mentioned, is that we do not know how to find back the minimum value of m from a tournament associated with a profile of m linear orders systematically. The second one deals with the parity of the weights of the arcs: there is no parity constraint in Problem 2. This point is not so important and can be relaxed. Indeed, if we consider twice each weight, we obviously obtain even weights, while the structures of the optimal solutions are not changed by this operation.

From Remark 1, it appears that we may consider Problem 2 instead of Problem 1. It is what we are going to do in the sequel. Though this problem appears in different contexts, for instance in electrical engineering, in agronomy, in biology, in mathematics, in economics, in archeology, in scheduling, in sports, in the social sciences, etc. (see for instance Barthélemy and Monjardet 1981, 1988; Grötschel et al. 1984a; Guénoche et al. 1994; Huber 1976; Jünger 1985; Lad et al. 2009; Lenstra 1977 or Reinelt 1985 for references or for applications in these fields), it will be with respect to the field of voting theory described above that we will denote or illustrate the entities defined or considered below.

Notation and basic definitions

By $T = (X, U)$, we will denote a tournament, and $p(x, y) \in \mathbb{N}$ will be the weight of the *arc* (x, y) (i.e., a directed edge; similarly, a *circuit* will denote a directed cycle). The number of vertices $|X|$ will be denoted by n. For any arc (x, y) of T, we say that x *beats* y or that y *is beaten by* x. A vertex x of T is a *Condorcet winner* (respectively *loser*) of T if x beats (respectively is beaten by) all the other vertices. Notice that such a Condorcet winner may not exist. If it exists, then it is unique. The same for a Condorcet loser. The (Copeland) *score* $s(x)$ of a vertex x is its out-degree, i.e., the number of vertices beaten by x. If n is odd and if all the scores are equal to $(n - 1)/2$, we say that the tournament is *regular*. If n is even and the maximum difference between any pair of scores is equal to 1, we say that the tournament is *quasi-regular* or *near-regular*. In this case, half the scores are equal to $(n/2) - 1$, and the other half are equal to $n/2$. For a subset V of U, we set $P(V) = \sum_{(x,y)\in V} p(x, y)$ and we call this quantity the *weight* of V. The transitive tournament obtained by reversing in T a subset V of arcs with a minimum weight $P(V)$ will be called a *median order* of T, or a *Kemeny order* if T is weighted, or a *Slater order* otherwise. When we reverse a subset V of arcs in T to obtain a linear order O (not necessarily a median order), we set $\kappa(T, O) = P(V)$ $(= \Delta(\Pi, O))$ and we call this quantity the *remoteness* of O from T. The minimum of $\kappa(T, O)$ over the set of the linear orders O defined on X will be denoted $K(T)$ and will be called the *Kemeny index of* T. A Kemeny order is thus a linear order of which the remoteness from T is equal to the Kemeny index of T. If all the weights of T are equal to 1, we say that T is *unweighted*. In an unweighted tournament T, the weight of a subset V of U is then equal to its cardinality. Thus, the search of a median order, then called a *Slater order*, corresponds to the search of a minimum number of arcs to be reversed in T to make T transitive. This minimum number of arcs is denoted $i(T)$ and is called the *Slater index of* T. A linear order $O = (X, W)$ defined on X is also written, for an appropriate labelling of the vertices, $x_1 > x_2 > \cdots > x_{n-1} > x_n$ with $(x_j, x_k) \in W$ if and only if the index j is smaller than the index k. In this case, we shall say also that x_j *is before* x_k. A part $x_1 > \cdots > x_{j-1} > x_j$ of

O with $1 \leq j \leq n$ will be called a *beginning section of* O. The vertex x_1 will be called the (Condorcet) *winner* of O. A vertex x will be called a *Kemeny winner* (respectively a *Slater winner*) of T if there exists a Kemeny order (respectively a Slater order) of T of which x is the winner. Kemeny losers and Slater losers can be defined in the same way, with respect to Condorcet losers of Kemeny orders or of Slater orders. Last, for any subset $V \subset U$, $\bar{V}$ will denote the set of the reversed arcs of V : $\bar{V} = \{(x, y)$ such that $(y, x) \in V\}$.

The paper is organized as follows. Based on equivalent statements for Problem 1, we study the complexity of the linear ordering problem and of the aggregation of linear orders or tournaments into a linear order in Sect. 2. Section 3 is devoted to the properties of the Slater winners or of the Kemeny winners. We investigate also some links with other tournament solutions in this section. Bounds for the Slater index can be found in Sect. 4, while the same is done in Sect. 5 for bounds for the Kemeny index. We make an inventory of methods allowing to solve Problem 1 approximately or exactly in Sect. 6. In the last section, we give partial answers to the following questions: how many Slater orders may a given tournament own, and how different can two Slater orders of a same tournament be?

2 Equivalent problems and complexity

As said above, Problem 2 consists in making T transitive by reversing a subset of arcs with a minimum total weight. It is well-known (see Moon 1968 for instance) that a tournament is transitive if and only if it does not contain any circuit. Moreover, a tournament does not contain any circuit if and only if it does not contain any circuit of length 3, i.e., a circuit going through 3 vertices and 3 arcs (such a circuit will be called a *3-circuit* in the sequel). Indeed, as a tournament T is a complete oriented graph, it is easy, from a given circuit C of T, to show the existence of a 3-circuit of T by considering the chords of C. The equivalence between transitive tournaments (or equivalently linear orders) and tournaments without circuit leads to Problem 3:

Problem 3 (*Minimum weighted feedback arc set problem*) Given a tournament $T = (X, U)$ weighted by a function p defined from U into $\mathbb{N}$, determine a subset V of U such that $P(V)$ is minimum and such that the graph $(X, U - V)$ obtained by removing from T the arcs of V is without circuit.

A *feedback arc set* is a subset of arcs such that removing all its elements deletes all the circuits of T. Equivalently, it is a subset of arcs such that any circuit of T contains at least one of these arcs (for a survey on feedback sets, see Festa et al. 1999). Thus Problem 3 consists in determining a feedback arc set with a minimum weight. It has been known for a long time (Karp 1972) that the computation of a minimum feedback arc set is an NP-hard problem for general directed graph (for a global presentation of the algorithmic complexity theory, see Ausiello et al. 2003; Barthélemy et al. 1996; Garey and Johnson 1979 or Hemaspaandra 2000). The complexity status of this problem for tournaments remained open for a long time, though several authors conjectured quite soon that it remains NP-hard for this kind of graphs. Recent results (see Alon 2006; Charbit et al. 2007 and Conitzer 2006) show that Problem 3 remains NP-hard when all the weights are equal to 1 (see also Ailon et al. 2005, where it is shown that this problem is NP-hard under randomized reductions):

Theorem 3 *The following problem is NP-complete*:

- *Name*: *Feedback arc set problem for tournaments.*
- *Instance*: *a tournament $T = (X, U)$, an integer K.*
- *Question*: *does there exist a subset V of U with $|V| \leq K$ and such that the graph $(X, U - V)$ obtained by removing the arcs of V from T is without any circuit?*

The parameterized complexity (see Downey and Fellows 1999 for a global presentation of this field) of the feedback arc set problem applied to tournaments has also been studied. Raman and Saurabh (2006) (see also Dom et al. 2006, where the case of bipartite tournaments is also considered) show that the feedback arc set problem for weighted or unweighted tournaments is fixed-parameter tractable (FPT) by providing appropriate algorithms. (Remember that a problem is said to be FPT if there exist an arbitrary function f and a polynomial function Q such that, for any instance (I, k) where I is an instance of the non-parameterized version of the problem and k is the considered parameter, (I, k) can be solved within a CPU time upper-bounded by $f(k)Q(|I|)$, where $|I|$ denotes the size of I. In particular, if k is upper-bounded by a constant, then the problem becomes polynomial.) More precisely, the parameterized version dealt with in Raman and Saurabh (2006) consists in considering the decision problem associated with Problem 3: the (weighted or unweighted) tournament T plays the role of I, and we introduce an integer k which will upper-bound the weight of the searched subset V. Then we set the question: does there exist a feedback arc set V of T with a weight—or cardinality if T is unweighted—less than or equal to k (if T is unweighted, we find back the problem stated in Theorem 3, with $k = K$)? Raman and Saurabh (2006) give several algorithms to solve the parameterized version of the feedback arc set problem applied to tournaments (they also consider the feedback vertex set problem as well as other kinds of graphs). The best complexity of their algorithms for this problem is $O(2.415^k n^\omega)$, where ω denotes the exponent of the running time of the best matrix multiplication algorithm (for instance, in the method designed by Coppersmith and Winograd (1990), ω is about 2.376). They also provide a $O(k4^k + \mu)$-algorithm for the parameterized version of the maximum acyclic subgraph problem (see Problem 4 below) for general directed graphs, where μ denotes the number of arcs of the considered graph ($\mu = n(n-1)/2$ for tournaments). This shows that this problem is also FPT, even for directed graphs which are not tournaments.

The interested reader will find other complexity results or questions about the approximability of these problems in the subsection "Approximation and probabilistic algorithms" of Sect. 6.1.

Moreover, it is easy to show that the optimal solutions of Problems 2 and 3 admit the same optimal weight, and that these optimal solutions differ from each other only by arcs with a weight equal to 0. Theorem 4 below specifies this property by generalizing to weighted tournaments a result stated by Younger (1963) for unweighted tournaments, and rediscovered by several authors (see also Barthélemy et al. 1995; Bermond 1972; Grindberg and Dambit 1965; Remage and Thompson 1964, 1966).

Theorem 4 *Problems 2 and 3 admit the same optimal value. Moreover, any optimal solution of Problem 2 is an optimal solution of Problem 3. Conversely, from any optimal solution V_3 of Problem 3, we may construct an optimal solution V_2 of Problem 2* (*with $P(V_2) = P(V_3)$*) *by the elimination of appropriate* 0*-weighted arcs from V_3.*

Remark 2 In fact, it is clear that any feasible solution of Problem 2 (i.e., any subset of arcs of which the reversing in T makes T transitive) is also a feasible solution of Problem 3 (a subset of arcs of which the deletion in T gives a graph without circuit). The converse is generally false, even if we only consider optimal solutions (because of 0-weighted arcs). For instance, removing all the arcs of a circuit does delete this circuit. But reversing all the arcs of a circuit creates another circuit (its reverse).

Because of this link between Problems 2 and 3, the complexity of Problem 2 follows as a corollary of Theorem 3:

Corollary 1 *Problem 2 is NP-hard, and remains so even if all the weights are equal to* 1.

In its turn, Corollary 1 yields the complexity of Problem 1, because of the polynomiality of the construction of a profile from a weighted tournament (see above). We may prove the following corollary (see Hudry 2008, 2010 for more detailed results; see Hudry 2008 and Wakabayashi 1986, 1998 for other results dealing with the complexity of the aggregation of binary relations; see Hudry 2009b for the complexity of other voting procedures):

Corollary 2

1. *Let m be any given positive integer. Then Problem 1 is NP-hard for a profile of m tournaments.*
2. *Let m be a positive integer large enough with respect to n ($m = \Omega(\sum_{m_{xy}>0} m_{xy})$ is sufficient when the weights are not all equal to* 0). *Then Problem 1 is NP-hard for a profile of m linear orders.*

Anyway, we may be more specific for the aggregation of an even number m of linear orders into a linear order. It is shown in Dwork et al. (2001) that Problem 1 is NP-hard when the profile contains 4 linear orders. By adding $(m-4)/2$ times a linear order O and the reversed order $\bar{O}$, it is easy to show Theorem 5. Notice that, because of the Paretian property (see Definition 4 and Theorem 18 below), Problem 1 is polynomial for a profile of 2 linear orders.

Theorem 5 *Let m be any given even integer with $m \geq 4$. Then Problem 1 is NP-hard for a profile of m linear orders.*

We may summarize the previous results as follows: Kemeny's problem is NP-hard (what was shown several years ago for n large enough: see for instance Bartholdi et al. 1989; Hudry 1989 or Orlin 1981), even for any given even number $m \geq 4$ of linear orders, and Slater's problem is also NP-hard (what are more recent results). Notice that the associated decision problems are NP-complete. We may derive Theorem 6 from the complexity of Slater's problem (see Hudry 2010) and from the one of Kemeny's problem (see Hudry 2009c for the proofs). The relationship between some of the following problems may seem obvious, and indeed it is the case for some of them. Anyway, remember that for another tournament solution, namely, the Banks solution—see Sect. 3—, checking that a given vertex is a Banks winner is an NP-hard problem (Woeginger 2003) while finding a Banks winner is polynomial (Hudry 2004) (see Hudry 2009a for a survey on the complexity of computing winners with respect to other tournament solutions).

Theorem 6 *Let T be a tournament (unweighted if we deal with Slater's problem, weighted if we deal with Kemeny's problem). The following problems are NP-hard:*

1. *computing the Slater index $i(T)$ of T,*
2. *computing a Slater winner of T,*
3. *computing a Slater order of T,*
4. *checking that a given vertex of T is a Slater winner of T,*
5. *computing the Kemeny index $K(T)$ of T,*
6. *computing a Kemeny winner of T,*
7. *computing a Kemeny order of T,*
8. *checking that a given vertex of T is a Kemeny winner of T.*

Theorem 5 gives the minimum even number m of linear orders so that Kemeny's problem is NP-hard. The same for m odd remains open:

Open problem 1 Determine the minimum odd value of m such that the aggregation of m linear orders into a linear order is NP-hard.

There exist other equivalent formulations for the problem of median orders or for Slater's problem. We give three of them. The equivalence between Problems 1, 2 or 3, and Problem 4 is obvious (with respect to the theory of complexity but not, for instance, with respect to the theory of approximation; see below) and, as said by Bermond (1972), its main interest is to link problems often studied separately.

Problem 4 (*Maximum acyclic subgraph problem*—notice that *maximum acircuitic* or *circuitless subdigraph problem* would be more appropriate, but this name is not used in literature) Given a weighted tournament T, find a partial graph of T without circuit and with a maximum weight.

It is clear that, if V is an optimal solution of a tournament $T = (X, U)$ for Problem 3, then $U - V$ is an optimal solution of T for Problem 4, and conversely.

To state Problem 5, we define the *weight-matrix* of a tournament T weighted by a function p as the matrix $M = (m_{ij})_{1 \le i \le n, 1 \le j \le n}$ with $m_{ij} = p(i, j)$ if the arc (i, j) exists in T and 0 otherwise. Notice that, for the existing arcs, it is in fact the same m_{ij} as the one introduced above in order to take the intensities of the preferences into account in the voting problem.

Problem 5 (*Optimal triangulation of a weight-matrix*) Given the weight-matrix M of a weighted tournament, determine a (same) permutation defined on the rows and on the columns of M so that the sum of the weights located under the diagonal is minimum.

This formulation was studied by Slater (1961) and by Younger (1963) for unweighted tournaments. A variant, sometimes known as *the triangulation of input-output matrices* (see for instance Grötschel et al. 1984b) consists in maximizing the sum of the weights located above the diagonal of the weight-matrix.

For the last formulation (used for instance in Bermond 1972 when T is unweighted), we define the weighted hypergraph $H(T)$ of the circuits of the tournament T. The vertex-set of $H(T)$ is given by the arcs of T belonging to at least one circuit of T. An edge of $H(T)$ corresponds to a circuit of T and conversely. Each vertex of $H(T)$ is weighted by the weight of the arc which it is associated with. Figure 1 illustrates such a hypergraph in the case of an unweighted tournament T; in $H(T)$, the arc (x, y) of T is noted xy.

Remind that a *vertex cover* of a hypergraph H is a subset of vertices such that each edge of H contains at least one element of this subset (see Berge 1985). If H is weighted, the

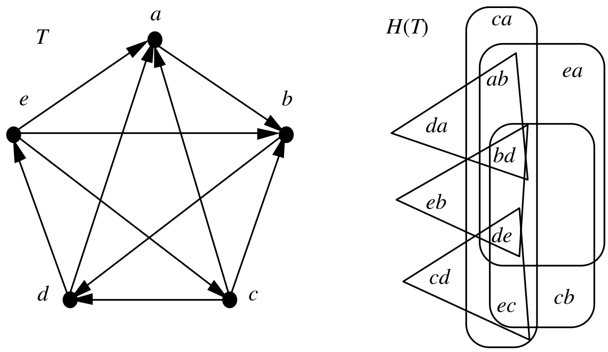


Fig. 1 A tournament T and the hypergraph $H(T)$ of the circuits of T

weight of a vertex cover of H is the sum of the weights of the vertices belonging to this vertex cover. For $H(T)$, it means that a vertex cover is a subset of arcs of T such that any circuit of T goes through at least one of these arcs, or, in other words, is a feedback arc set of T. Hence the equivalence between Problem 3 and Problem 6 below.

Problem 6 Given a weighted tournament T, determine a vertex cover of $H(T)$ with a minimum weight.

Consequently, in the case of an unweighted tournament T, the Slater index $i(T)$ of T is equal to the minimum cardinality of a vertex cover of the associated hypergraph $H(T)$ (this equality appears already in Bermond 1972). Hence, for the example of Fig. 1, as the vertices *bd* and *ec* of $H(T)$ cover all the edges of $H(T)$ while no vertex of $H(T)$ can cover alone all the edges of $H(T)$, we have $i(T) = 2$ and reversing the arcs (b, d) and (e, c) in T leads to a Slater order (it is not the only one) $c > d > e > a > b$.

Because of the equivalence between the previous six problems and because of Theorem 3, all of them are NP-hard:

Theorem 7 *Problems 1 to 6 are NP-hard.*

3 Properties of the winners and links with other tournament solutions

As polynomial algorithms are not known to compute a Kemeny order or a Slater order, many exact methods have been designed that are more or less based on an exhaustive enumeration of the possible linear orders (see below). Then it can be interesting to know properties fulfilled by median orders in order to limit the enumeration. This section is devoted to some of these properties (see also Laslier 1997 for axiomatic properties verified or not by Slater orders) and to the relationships between Slater winners or Kemeny winners and the winners according to other procedures.

Most of the following properties give necessary conditions for a vertex to be a Kemeny or a Slater winner, but the first one (due to Younger 1963 for unweighted tournaments and to Jacquet-Lagrèze 1969 for weighted tournaments) and its corollaries deal with a global property of medians orders.

Theorem 8 *Let T be a tournament and let $O = x_1 > x_2 > \cdots > x_{n-1} > x_n$ be a median order of T. Then, for any i and any j with $i < j$, $x_i > x_{i+1} > \cdots > x_{j-1} > x_j$ is a median order of the subtournament induced by the vertices $x_i, x_{i+1}, \ldots, x_{j-1}, x_j$.*

By considering the special case $j = i + 1$ in the statement of Theorem 8, we obtain the following corollaries.

Corollary 3 *Let $O = x_1 > x_2 > \cdots > x_{n-1} > x_n$ be a median order of a tournament $T = (X, U)$ weighted by p. Then, for any i with $1 \le i < n$, we have: $(x_i, x_{i+1}) \in U$ or $p(x_{i+1}, x_i) = 0$.*

In other words, let $O = x_1 > x_2 > \cdots > x_{n-1} > x_n$ be a median order of a weighted tournament $T = (X, U)$. If the series of arcs $(x_i, x_{i+1})_{1 \le i < n}$ does not define a Hamiltonian path of T, then the "badly oriented" arcs all have a weight equal to 0. Notice that reversing these "badly oriented" arcs in O gives another median order of T which is associated with a Hamiltonian path of T. So, there always exists a median order with which we may associate a Hamiltonian path of T. Moreover, if all the weights are positive (it is the case for instance if we aggregate an odd number of linear orders into a linear order), then any median order is necessarily associated with a Hamiltonian path of T. A special case (stated by Remage and Thompson 1966) is the one of Slater orders (see also Bar-Noy and Naor 1990 for a link between Hamiltonian paths and minimal—with respect to inclusion—feedback arc sets in tournaments):

Corollary 4 *Any Slater order $x_1 > x_2 > \cdots > x_{n-1} > x_n$ of a tournament T is such that the arcs (x_j, x_{j+1}) for $1 \le j < n$ define a Hamiltonian path of T.*

These properties allow to prune the search tree of a branch and bound method (it is done for instance in Charon and Hudry 2006). Another way to prune the search tree is given by Theorem 9 below. For this theorem, we need new definitions and a lemma.

Definition 1

1. Let Y be a subset of X and let x belong to X. We call *partial weight of x with respect to Y* the quantity $P_Y(x) = \sum_{y \in Y} p(x, y)$ with the following understanding: if (x, y) is not an arc of the considered tournament, we set $p(x, y) = -p(y, x)$ (and $p(x, x) = 0$).
2. Let O be a linear order defined on q elements of X for some integer q ($q \le n$). We call *shift of parameters i and j (with $i \ne j$) applied to O* the operation consisting in changing O into another linear order O' obtained by pushing on the left if $i < j$ (respectively on the right if $i > j$) all the vertices located in O in the positions $i + 1, i + 2, \ldots, j$ (respectively $j, j + 1, \ldots, i - 1$) and then by putting at the position j in O' the vertex which was at the position i in O. In other words, for $i < j$, if we have:

$$O = x_1 > \cdots > x_{i-1} > x_i > x_{i+1} > \cdots > x_{j-1} > x_j > x_{j+1} > \cdots > x_q,$$

a shift of parameters i and j gives the order:

$$x_1 > \cdots > x_{i-1} > x_{i+1} > \cdots > x_j > x_i > x_{j+1} > \cdots > x_{q-1} > x_q,$$

while a shift of parameters j and i gives the order:

$$x_1 > \cdots > x_{i-1} > x_j > x_i > \cdots > x_{j-1} > x_{j+1} > \cdots > x_{q-1} > x_q.$$

Lemma 1, whose proof is straightforward, specifies the variations of κ when a shift is performed (see for instance Charon et al. 1997a).

Lemma 1 *Let T be any tournament weighted by p, and let O be the order $x_1 > \cdots > x_n$. Applying a shift of parameters i and j $(i \neq j)$ to O gives an order O' with*:

1. $\kappa(T, O') - \kappa(T, O) = P_{\{x_{i+1},\ldots,x_j\}}(x_i)$ *if* $i < j$,
2. $\kappa(T, O') - \kappa(T, O) = -P_{\{x_j,\ldots,x_{i-1}\}}(x_i)$ *if* $i > j$.

We may apply Lemma 1 to prove Theorem 9, which provides a property of a beginning section of a median order (see for instance Charon et al. 1997a).

Theorem 9 *Let $S = x_1 > x_2 > \cdots > x_{q-1} > x_q$ be a beginning section of an order defined on X. If S is the beginning section of a median order of T, then we have*:

1. $P_Y(x_q) \geq 0$, *with* $Y = X - \{x_1, x_2, \ldots, x_{q-1}\}$,
2. *for any i with* $1 \leq i \leq q-1$, $P_{\{x_i,\ldots,x_{q-1}\}}(x_q) \leq 0$ *and* $P_{\{x_{i+1},\ldots,x_q\}}(x_i) \geq 0$.

An interesting case of Theorem 9 is the one with $q = 1$: for a vertex x to be a winner, the sum of the weights of the arcs leaving x must be greater than or equal to the sum of the weights of the arcs entering x. This is a generalization of a result applied by Bermond (1972) to unweighted tournaments: if x is a Slater winner, then the score $s(x)$ of x is greater than or equal to $(n-1)/2$. With this respect, we may be more specific:

Theorem 10 *A tournament T is regular* (*hence with an odd number n of vertices*) *if and only if every vertex of T is a Slater winner. If T is not regular, then there exists at least one Slater winner of which the score $s(x)$ is greater than or equal to $n/2$.*

Notice that, while it is generally difficult to compute a Slater winner (see Theorem 6), the computation of all the Slater winners for regular tournaments is trivial, but the one of their Slater indices is not! We can even wonder what the complexity status of this problem is when restricted to this family of tournaments:

Open problem 2 What is the complexity status of the computation of $i(T)$ when T is regular?

If we come back to the initial voting problem and with the same notation, Theorem 9 gives a result shown by Monjardet (1973): if x is a winner, then we have $\sum_{y \neq x} \sum_{j=1}^{m} t_{xy}^{j} \geq \lceil \frac{(n-1)m}{2} \rceil$ (in other words, if x is a winner, the number of votes in favour of x must be greater than or equal to the number of votes in disfavour of x in the paired-comparisons). These results have been extended by Guénoche (1995) who, from a heuristic solution and the weights of the arcs of T, gives bounds for the rank of any vertex in an optimal solution.

The second part of Theorem 9, of which it is easy to take advantage from an algorithmic point of view (it can be implemented with an amortized complexity of $O(n)$ with a memory space of $O(n)$), characterizes the linear orders O which cannot be improved by shifts: it will be the case if all the beginning sections of O fulfil the condition. By the way, we may notice that this second part of Theorem 9 gives back Corollary 3 and the Hamiltonian principle: it is sufficient to consider the case $i = q-1$ in the first inequality of the second part (it gives also the first condition of the same Theorem 9, but it is interesting to distinguish them from an algorithmic point of view).

It could be tempting to infer from Theorem 9 that a Kemeny winner is a vertex x maximizing $P_X(x)$, i.e., maximizing the difference between the sum of the weights of the arcs

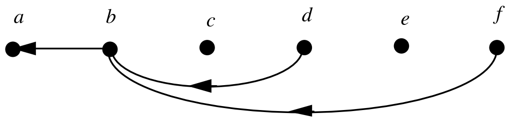


Fig. 2 A minimum tournament for which the set of Slater winners and the set of Copeland winners are disjoint

of type (x, y) and the sum of the weights of the arcs of type (z, x). But such a conclusion is false, even if all the weights are equal to 1. Notice that, when all the weights are equal to 1, $P_X(x)$ is equal to $2s(x) - 1$, and so, maximizing $P_X(x)$ is the same as maximizing the score $s(x)$. Ranking the vertices according to their decreasing scores is a tournament solution suggested by Copeland (1951) and a vertex with a maximum score is called a *Copeland winner* (see also Laslier 1997 for the properties of this tournament solution). Bermond (1972) gives a tournament with 7 vertices such that no Slater winner is a Copeland winner.

We can be more specific. In the tournament of Fig. 2, for which the missing arcs are assumed to be oriented from the left to the right, it is easy to show that b is the only Slater winner while its score is not the highest one (see Charon et al. 1996c for more details). On the other hand, we may check easily (for instance from the list of the different patterns of non-isomorphic tournaments given in Moon 1968) that:

- there always exists a Slater winner which is also a Copeland winner if the number n of vertices is equal to 5,
- all the Slater winners are Copeland winners if n is equal to 4,
- the set of Slater winners and the one of Copeland winners are the same if n is less than or equal to 3.

This involves the same conclusions with respect to the maximum likelihood method due to Zermelo (1929), since this method leads to the same optimal orders as Copeland's solution (see for instance Laslier 1997).

Copeland's method provides orders which are not "too far" from an optimal solution. More precisely, consider the ratio between, on one hand, the distance between the tournament T and a Copeland order of T and, on the other hand, the distance between T and a Slater order of T (see the subsection "Approximation and probabilistic algorithms" of Sect. 6.1); then this ratio does not exceed 5. But the distance between a Copeland order of T and a Slater order of T can be very large, as shown in Charon and Hudry (2009c):

Theorem 11 *Let n be an integer greater than or equal to* 3.

- *If $\xi(n)$ denotes the maximum distance between a Copeland order of a tournament T on n vertices and a Slater order of the same tournament T, then*:

$$\xi(n) = n(n-1)/2 \quad \textit{for } n \textit{ odd} \quad \textit{and} \quad \xi(n) = (n^2 - 3n + 2)/2 \quad \textit{for } n \textit{ even.}$$

- *If $\zeta(n)$ denotes the maximum distance between a Copeland order of a strongly connected tournament T on n vertices and a Slater order of the same tournament T, then*:

$$\zeta(n) = n(n-1)/2 \quad \textit{for } n \text{ odd} \quad \textit{and} \quad \eta(n) = (n^2 - 3n - 2)/2 \quad \textit{for } n \textit{ even with } n \geq 8.$$

- *If $\eta(n)$ denotes the maximum distance between a Copeland order of a tournament T on n vertices admitting only one Slater order and this Slater order of T, then*:

$$\eta(n) = (n^2 - 5n + 6)/2 \quad \textit{for } n \textit{ odd} \quad \textit{and} \quad \eta(n) = (n^2 - 5n + 8)/2 \quad \textit{for } n \textit{ even.}$$

Among the other tournament solutions, one (proposed by Fishburn 1977 and by Miller 1980; see also Laslier 1997) consists in keeping the *uncovered vertices* of an unweighted tournament T as the winners of T. The following definition generalizes the usual concept of covering to weighted tournaments.

Definition 2 Let x and y be two distinct vertices of T. We say that x *covers* y if we have: $\forall z \in X$, $p(x,z) \geq p(y,z)$ (with the same agreement as in Definitions 1 for the arcs which do not exist in T) and $\exists z \in X$ with $p(x,z) > p(y,z)$. A vertex is *uncovered* if no vertex covers it; $UC(T)$ will denote the set of the uncovered vertices of T, and $T_{|UC(T)}$ will denote the subtournament of T induced by the uncovered vertices of T.

Notice that, unless the weight of the arc between the two vertices is equal to 0, if x covers y, then x also beats y.

The following theorem is straightforward:

Theorem 12 *The covering relation defines a partial order* (*called the trace order*) *on* X, *of which the maximal elements are the uncovered vertices of* T.

Remark 3 In the case of unweighted tournaments, x covers y if and only if x beats y and every vertex beaten by y is also beaten by x, which is the definition of the usual covering relation. In this case, any uncovered vertex is a *centre* (also called a *king*) of T and conversely. Remember that, in a tournament, a centre x is a vertex which reaches any other vertex y through one or two arcs: $(x,y) \notin U \Rightarrow \exists z \in X$ with $(x,z) \in U$ and $(z,y) \in U$. In particular, every Copeland winner is a centre, but not conversely.

To specify the links between uncovered vertices in weighted or unweighted tournaments and Kemeny or Slater winners, we need another definition. Assume that T is not necessarily strongly connected and consider its q $(q \geq 1)$ strongly connected components $C_1, C_2, \ldots, C_q$. The relation $>_{\mathrm{scc}}$ defined on the components C_i $(1 \leq i \leq q)$ by $C_i >_{\mathrm{scc}} C_j$ (with $i \neq j$) if and only if there exist $x_i \in C_i$ and $x_j \in C_j$ with $(x_i, x_j) \in U$ is a linear order. So there exists a unique maximal element with respect to $>_{\mathrm{scc}}$ (which is T itself if T is strongly connected). The next definition specifies this element.

Definition 3 The strongly connected component of T which is the maximal element with respect to $>_{\mathrm{scc}}$ is called the *top cycle of* T. It will be denoted by $TC(T)$.

The orientation of the arcs with only one extremity in $TC(T)$ is the same for all these arcs: they go from $TC(T)$ to the outside of $TC(T)$. Theorem 13 (proved in Charon et al. 1997a with the help of appropriate shifts) generalizes to weighted tournaments a result due to Banks et al. (1991) for unweighted tournaments.

Theorem 13

1. *There exists a Kemeny winner which belongs to* $TC(T_{|UC(T)})$. *If all the weights are strictly positive*, *all the Kemeny winners of* T *belong to* $TC(T_{|UC(T)})$.
2. *If the weights are all strictly positive*, *any median order is a linear extension of the trace order of* T. *Otherwise*, *there exists at least one median order which is a linear extension of the trace order of* T.
3. *All the Slater winners belong to* $TC(T_{|UC(T)})$. *All the Slater orders of* T *are linear extensions of the trace order of* T.

Remark 4

1. As $TC(T_{|UC(T)})$ is included in $TC(T)$ and in $UC(T)$, the results of Theorem 13 can be adapted by replacing $TC(T_{|UC(T)})$ with $TC(T)$ or with $UC(T)$ (such a result is weaker, but sometimes easier to apply).
2. The difficulty due to the potential existence of 0-weighted arcs is not so important as it may seem, at least for the computation of median orders. A weight equal to 0 for an arc (x, y) shows that the arc (y, x) is as legitimate as (x, y). If we apply the criteria that we should normally apply only for positive weights while there are weights equal to 0, some median orders may not be computed. It is anyway easy to reconstruct these forgotten median orders from the computed ones. Obviously, this difficulty does not occur for Slater's problem.
3. At least some Kemeny winners and all Slater winners are uncovered vertices. Anyway, it may happen that no Kemeny winner and no Slater winner belongs to the set $UC^2(T) = UC(T_{|UC(T)})$ of the uncovered vertices of the subtournament of T induced by the uncovered set of T. (By the way, this shows that the covering relation is not so trivial.) A tournament with 8 vertices illustrating this situation can be found in Laffond et al. (1991a).

The following result is proved in Barthélemy et al. (1995).

Theorem 14 *In an unweighted tournament T, the arcs of which the orientations are reversed in T to obtain a Slater order all belong to* 3*-circuits.*

This result can be related to the covering relation, as stated by Theorem 15, whose proof is left to the reader.

Theorem 15 *In an unweighted tournament T, an arc (x, y) of T belongs to a* 3*-circuit of T if and only if y is not covered by x in T.*

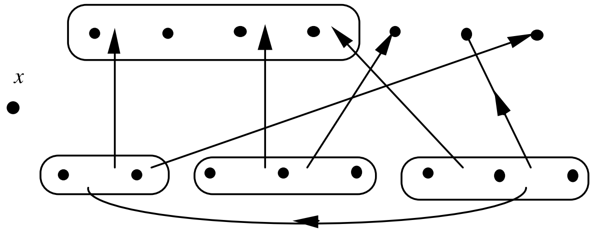
With this respect, Theorem 14 says that the arc (x, y) will not be reversed if x covers y. This is the same as saying that a Slater order is a linear extension of the trace order of T. Similarly, if x covers y in T, then Theorem 13 says that x is ranked before y in any Slater order of T: so, (x, y) cannot be reversed, which means that only the arcs (x, y) such that x does not cover y, i.e., arcs belonging to some 3-circuits, can be reversed.

The following theorem, of which the easy proof is left to the reader, is useful when the considered tournament is not strongly connected.

Theorem 16 *Let T be a tournament which is not strongly connected and let $C_1, C_2, \dots, C_q$ be its strongly connected components with the assumption that all the arcs between C_i and C_j are oriented from C_i to C_j if and only if i is less than j. Let $O_1, O_2, \dots, O_q$ be median orders of the subtournaments induced respectively by $C_1, C_2, \dots, C_q$. Then*:

1. *the linear order $O_1 > O_2 > \cdots > O_q$ obtained by the concatenation of $O_1, O_2, \dots, O_q$ is a median order of T*,
2. *if all the weights are strictly positive*, *any median order of T is a concatenation of the previous form.*

Fig. 3 A 16-vertex tournament for which the set of Slater winners and the one of Banks winners are disjoint



Remark 5

1. If the sorted scores of the vertices of T (weighted or not) are known, it is possible to compute the strongly connected components of T in $O(n)$, thanks to the characterization of the scores given by Landau (1953) (more precisely, we may compute $TC(T)$ in $O(|TC(T)|)$).
2. When some weights are equal to 0, indications are given in Charon et al. (1997a) about the median orders of T which are not obtained as a concatenation of median orders of the subtournaments induced by the strongly connected components of T.

The next theorem shows that Slater winners and *Banks winners* (associated with the tournament solution designed by Banks 1985) can be distinct. A *Banks winner* of T is the Condorcet winner of a maximal (with respect to inclusion) transitive subtournament of T (see Laslier 1997 for properties of this tournament solution). Laffond and Laslier (1991) provide a tournament with 75 vertices such that the set of Copeland winners, the set of Slater winners and the set of Banks winners are pairwise disjoint. Theorem 17 gives a tournament with 16 vertices (taken from Charon et al. 1997b) such that the set of Banks winners and the set of Slater winners are disjoint, and a tournament with 13 vertices (taken from Hudry 1998) such that there exists a Slater winner which is not a Banks winner.

In order to make the tournaments easier to read, we do not draw all the arcs. In Fig. 3, the vertices are gathered in clusters (sometimes called *homogeneous parts*, or *intervals*, or *sets of similar vertices*...). An arc from a cluster A towards a cluster B means that, for any vertex a of A and any vertex b of B, the arc between a and b is oriented from a to b. Moreover, all the missing arcs are oriented from the left to the right between vertices belonging to a same horizontal layer, or from the top to the bottom for vertices located on different layers. Similarly, in Fig. 4, the missing arcs are oriented from the left to the right or from the top to the bottom.

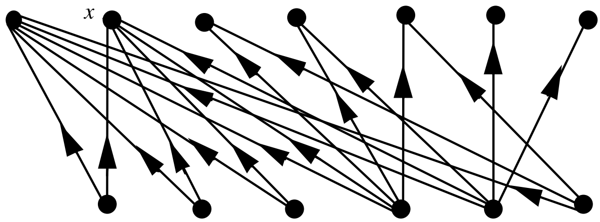
Theorem 17

1. *In the tournament of Fig.* 3, x *is the only Slater winner but it is not a Banks winner.*
2. *In the tournament of Fig.* 4, x *is a Slater winner but it is not a Banks winner.*

The problem of a minimum tournament fulfilling these properties remains open.[2] Notice that the problem of a minimum tournament with a Banks winner which is not a Slater winner

[2]During the publication process, Östergård and Vaskelainen (2009) showed that there exists a tournament on 14 vertices such that the set of Slater winners and the one of Banks winners are disjoint; moreover, they show, thanks to a study based on a computer, that 11 is the minimum order of a tournament in which the Slater set is not a subset of the Banks set.

Fig. 4 A 13-vertex tournament with a Slater winner (x) which is not a Banks winner



is trivial: it is the case for the (unique, up to an isomorphism) strongly connected tournament defined on four vertices.

Open problem 3 What is the minimum number of vertices of a tournament without common Slater winner and Banks winner?

Other tournament solutions can be found in Laslier (1997): the successive iterations UC^k ($k \geq 1$) of UC (more precisely, UC^k is defined by $UC^1 = UC$ and, for $k > 1$, $UC^k(T) = UC(T_{|UC^{k-1}(T)})$), the *minimal covering set* defined by Dutta (1988), which is a refinement of the successive iterations of UC, the *bipartisan set* defined by Laffond et al. (1991b) from the game-theoretical concept of Nash equilibrium and which is a refinement of the minimal covering set (see also Laffond et al. 1993), the *tournament equilibrium set* defined by Schwartz (1990) and which is also a refinement of the successive iterations of UC. All these solutions except UC^1 (=UC) select winners which belong to UC^2. So, there exist tournaments of which the set of Slater winners and any of these sets (but UC) are disjoint (see Remark 4). The same situation of an empty intersection of the sets of winners may occur with two other tournament solutions. One is called "long path method" in Laslier (1997). It was already suggested by Wei (1952) and then proposed by several authors (see Laslier 1997). It is based on the limit of the successive normalized powers of the adjacency matrix M of T or, equivalently, on the eigenvectors of M. The other one is called "Markov method" in Laslier (1997). It is based on the probabilities arising from a Markov chain defined on the vertices of T. Laslier (1997) gives a tournament on 7 vertices with only one Slater winner which is not a winner according to the long path method or to the Markov method.

Similarly, a Slater winner is not necessarily a Kemeny winner and conversely. What is more, it is easy to design tournaments with $n \geq 4$ such that the unique Slater winner is the unique Kemeny loser or such that the unique Kemeny winner is the unique Slater loser (see for instance Klamler 2003). Charon and Hudry (2009d) investigate the relationships between Slater's solutions and Kemeny's solutions and generalize Klamler's results. Similar results exist with respect to Dodgson method (Dodgson 1876). Remember that Dodgson method consists, when there is no Condorcet winner, in computing the fewest changes in voters' preferences necessary to create a Condorcet winner. Such a candidate is then called a *Dodgson winner* (the computation of Dodgson winners is also NP-hard; see Hudry 2009b for a survey of the complexity of several voting procedures). This concept can be extended to obtain a ranking of all the candidates. Klamler (2004) shows that, for $n \geq 3$, there exist profiles such that the unique Kemeny winner or the unique Slater winner can occur at any position in the Dodgson ranking. Ratliff (2001) proved also that, for $n \geq 4$, it is even possible to find situations such that a Kemeny winner can be found at any position in a Dodgson order and, conversely, that a Dodgson winner can be found at any position in a Kemeny order.

To conclude this section, we need a new definition.

Definition 4 Let $\Pi = (O_1, O_2, \ldots, O_m)$ be a profile of m linear orders. We call *unanimity* (*partial*) *order* or *Pareto* (*partial*) *order of* Π the partial order $\bigcap_{1 \le j \le m} O_j$ consisting of the intersection of all the linear orders O_j $(1 \le j \le m)$ of the profile.

Theorem 18, due to Monjardet (1973), shows that the median orders are linear extensions of the unanimity order (such a property is said to be *Paretian*; similarly, a median order of a profile of linear orders is said to be *Paretian*):

Theorem 18 *Any median order of a profile of linear orders is a linear extension of the unanimity order of this profile.*

So, if one of the arcs of the tournament representing Π has a weight equal to the number m of linear orders of Π (if this number is known, what is not obvious if we deal only with the weighted tournament representing Π), then such an arc cannot be reversed when a median order is computed. Notice anyway that the unanimity order can be empty.

4 Bounds of the Slater index

It is always possible to upper-bound the Slater index $i(T)$ of any given tournament T with the solution provided by a heuristic. We do not consider such a possibility here. We give upper or lower bounds available for any tournament. For this, let $I(n)$ be the maximum of the values that $i(T)$ can take when T is any tournament with n vertices. Notice that the minimum of $i(T)$ is trivial: this minimum is equal to 0 and is reached by transitive tournaments and only by them. It is also easy to show that all the values between 0 and $I(n)$ can also be reached.

It is quite obvious that $I(n)$ cannot be greater than half the number of arcs: otherwise, it would be better to reverse the arcs that we do not reverse! Anyway, Theorems 22 and 23 will show that $I(n)$ is not far from this value. The next theorem is due to Reid (1969) (this result is sharper for $n \ge 8$ than the one by Poljak and Turzík 1986: these authors notice that, for any n, it is always possible to build a linear order of which the distance from T is less than or equal to $(n-1)^2/4$).

Theorem 19 *For $n \ge 8$, we have $I(n) \le \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$.*

The following result, quoted in Bermond (1972) and due to Jung (1970) as well as Corollaries 5 and 6, gives a result still sharper, but also more difficult to exploit.

Theorem 20 *Let n_1 and n_2 be two integers with $n_1 + n_2 = n$ and $n_1 \le n_2$. Then*:

- *if n is even, $I(n) \le I(n_1) + I(n_2) + \frac{n_1(n_2-1)}{2}$,*
- *if n is odd, $I(n) \le I(n_1) + I(n_2) + \frac{n_1 n_2}{2}$.*

Corollary 5 *If there exists an integer k such that n is equal to 2^k, then*:

$$I(2^k) \le 2^{2k-2} - (k+1)2^{k-2} = \frac{n^2}{4} - (\log_2 n + 1)\frac{n}{4}.$$

Corollary 6 *Let n be an even integer*: $n = \sum_{i=1}^{q} 2^{k_i}$ *with* $0 < k_1 < k_2 < \cdots < k_q$. *Then*:

$$I(n) \le \frac{1}{4}n(n-1) - \frac{1}{4}\sum_{i=1}^{q}(k_i + 2q - 2i)2^{k_i}.$$

If n is odd, *then*:

$$I(n) \le I(n-1) + \frac{n-1}{2}.$$

In fact, these results come from another result which can be applied to any tournament. Let (X_1, X_2) be any bipartition of X. In order to make T transitive, it is sufficient to make the tournaments T_1 and T_2 induced from T respectively by X_1 and X_2 transitive and to reverse the $m^+(X_1, X_2)$ arcs oriented from X_1 towards X_2 or, conversely, the $m^+(X_2, X_1)$ arcs oriented from X_2 towards X_1. This result can be stated as follows:

Theorem 21 *For any tournament T and any bipartition (X_1, X_2) of X*, *we have*:

$$i(T) \le i(T_1) + i(T_2) + \min\{m^+(X_1, X_2); m^+(X_2, X_1)\},$$

where T_1 and T_2 are the subtournaments induced by X_1 and X_2 respectively.

Theorem 21 can immediately be generalized to the Kemeny index $K(T)$ of any weighted tournament T by considering, instead of their number, the sum of the weights of the arcs going from X_1 to X_2 or from X_2 to X_1, and by substituting $K(T_1)$ (respectively $K(T_2)$) to $i(T_1)$ (respectively $i(T_2)$).

The next theorem is due to Erdös and Moon (1965). It gives a lower bound of $I(n)$. Since, as noticed previously, $I(n)$ is trivially upper-bounded by $\lfloor n(n-1)/4 \rfloor$, it shows that the quantity $4I(n)/n^2$ tends asymptotically towards 1.

Theorem 22 *For any* $\varepsilon > 0$, *there exists* n_ε *such that* $n > n_\varepsilon$ *involves* $I(n) > \frac{n(n-1)}{4}(1-\varepsilon)$.

We may sharpen this bound, thanks to the results provided by probabilistic methods by Spencer (1971, 1978, 1987) and by de la Vega (1983) (see also Sect. 6). The next theorem gives sharper upper and lower bounds for $I(n)$.

Theorem 23 *For n large enough*, *we have*:

$$\left\lceil \frac{n(n-1)}{4} - \frac{7n\sqrt{n}}{4} \right\rceil \le I(n) \le \left\lfloor \frac{n(n-1)}{4} - \frac{n\sqrt{n}}{8\sqrt{\pi}} \right\rfloor.$$

Moreover, de la Vega (1983) proved that, still for n large enough, the value of $i(T)$ is greater than or equal to $\frac{n(n-1)}{4} - \frac{7n\sqrt{n}}{4}$ for almost all the tournaments T randomly generated as follows. For every pair $\{x, y\}$ of vertices, the orientation of the arc between x and y is randomly chosen with a same probability (hence equal to 0.5) for the two possibilities, the choice for any arc being independent of the choice of the other arcs. (More precisely, de la Vega and Spencer studied Problem 4, i.e., the search of a partial graph without circuit but with a maximum number of arcs. The link between Problem 4 and Slater's problem allows to adapt their results in order to meet Theorem 23.) By the way, this shows that this random generation, when used for instance to test or to compare methods designed to solve Slater's problem, may introduce a bias, because leading to tournaments with large Slater indices. The algorithm proposed in Charon et al. (1996a) aims at obviating such a drawback, and gives the possibility to randomly generate tournaments with given scores, in such a way that any tournament with the prescribed scores has a non-zero probability to be generated. Other random generations of tournaments can be found in Charon and Hudry (2006).

The exact values of $I(n)$ are given below for $n \leq 13$, then bounds for $14 \leq n \leq 31$ (they come from Bermond and Kodratoff 1976; Smith 2007; Woirgard 1997 and, for 30 and 31, by the application of the previous theorems; see also Bermond 1972, 1975; Hardouin Duparc 1975 and Kotzig 1975).

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$I(n)$	0	0	1	1	3	4	7	8	12	15	20	22	28

n	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
Lower bound	31	38	40	47	55	64	65	74	81	92	99	110	122	135	138	151	165	180
Upper bound	32	39	44	52	58	67	73	83	91	102	110	122	137	150	161	174	181	196

Theorem 24 comes from Thomassen (1975) (see also Bermond and Kodratoff 1976):

Theorem 24 *For any q and any r, we have*:

$$I(qr) \geq \mathrm{Max}\{qI(r) + r^2 I(q), rI(q) + q^2 I(r)\}$$

and

$$I(qr+1) \geq \mathrm{Max}\{qI(r+1) + r^2 I(q), rI(q+1) + q^2 I(r)\}.$$

Though the exact value of $I(n)$ and the structure of tournaments with a maximum Slater index are not known for every n, we may recall the conjectures stated by Bermond (1972):

Conjecture 1 *If n is odd, the tournaments with $I(n)$ as their Slater indices are regular.*

Conjecture 2 *If n is odd with $n \geq 3$, $I(n) = I(n-1) + (n-1)/2$.*

Conjecture 3 *For any n, there exists a regular (if n is odd) or a quasi-regular (if n is even) tournament with $I(n)$ as its Slater index.*

Remark 6

1. Conjecture 2 yields the other two.
2. Conjecture 1 is true for $n \leq 11$ (see Bermond 1972). On the other hand, Bermond (1972) gives a tournament on 8 vertices which is not quasi-regular but of which the Slater index reaches the maximum $I(8)$. This shows that the counterpart of Conjecture 1 for n even, i.e., the statement "if n is even, the tournaments with $I(n)$ as their Slater indices are quasi-regular", is false.
3. Conjectures 1 and 3 are reinforced by the result due to de la Vega (see above), since the random generation of tournaments with a probability equal to 0.5 for each possible direction of any arc leads to large Slater index and simultaneously to scores close to each other. Theorem 35 reinforces also Conjectures 1 and 3.

It is also interesting (for instance to design evaluation functions in branch and bound methods; see below) to have lower bounds of $i(T)$ for any tournament T. The following results (Bermond 1972) are related to the formulation of Problem 6, in which we look for a vertex cover of the hypergraph $H(T)$ of the circuits of T. Any (lower or upper) bound of the minimum cardinality of a vertex cover of $H(T)$ provides a (lower or upper) bound of $i(T)$. The two main drawbacks of such an approach rely on the fact that the computation of a minimum vertex cover is NP-hard (see Garey and Johnson 1979), and on the fact that $H(T)$ may be very large, with up to $n(n-1)/2$ vertices (when each arc belongs to a circuit; this situation occurs for instance for regular tournaments, for n odd, see Alspach 1967) and with a number of edges which can be exponential with respect to n. For instance, even if we consider only the Hamiltonian circuits, there are some regular tournaments which admit at least $2^{(n-3)/2}$ Hamiltonian circuits. So, from a practical point of view, it is quite often too long to build $H(T)$ explicitly. To obtain a lower bound of $i(T)$, we may restrict ourselves to the hypergraph $H_3(T)$ of the 3-circuits of T. This hypergraph is defined as $H(T)$, but by considering only the 3-circuits of T instead of all the circuits of T. The number of vertices of $H_3(T)$ may still be equal to $n(n-1)/2$ (see Alspach 1967), but its number of edges is equal to the number of 3-circuits of T. If the scores of T are $s_1, s_2, \ldots, s_n$, the number $C_3(T)$ of 3-circuits of T is given by (see Kendall and Babington Smith 1940):

$$C_3(T) = \frac{n(n-1)(n-2)}{6} - \sum_{j=1}^{n} \frac{s_j(s_j-1)}{2} \leq \begin{cases} (n^3-n)/24 & \text{if } n \text{ is odd,} \\ (n^3-4n)/24 & \text{if } n \text{ is even} \end{cases}$$

and the bounds are reached by regular (for n odd) or quasi-regular (for n even) tournaments.

By the way, notice that $C_3(T)$ is the exact number of arcs that must be reversed to make T transitive if we reverse only arcs of which the two extremities have the same score (Kadane 1966).

If we call $\tau_3(T)$ (respectively $\tau(T)$) the minimum cardinality of a vertex cover of $H_3(T)$ (respectively $H(T)$), we have, since $H_3(T)$ is a subgraph of $H(T)$ and because of the equality $\tau(T) = i(T)$:

Theorem 25 *For any tournament T, $\tau_3(T) \leq i(T)$.*

Nonetheless, the exact computation of $\tau_3(T)$ is usually still too long. So, we may usually work with a lower bound of $\tau_3(T)$, which hence will be also a lower bound of $i(T)$. In particular, we may adopt the following lower bound based on the degrees of $H_3(T)$, as done in Charon et al. (1997a), the degree of a vertex u of $H_3(T)$ (that is, an arc of T) being in fact the number of 3-circuits going through u in T.

Definition 5 Let $\partial_1, \partial_2, \partial_3, \ldots$ be the degrees of the vertices of $H_3(T)$ (i.e., some arcs of T), where the numbering is assumed to be done according to the decreasing values of these degrees: $\partial_1 \geq \partial_2 \geq \partial_3 \geq \cdots$. Parameter $\chi(T)$ is defined as the smallest integer χ verifying $\sum_{k=1}^{\chi} \partial_k \geq C_3(T)$.

Theorem 26 *For any tournament T, we have $\chi(T) \leq \tau_3(T)$.*

Another possibility to lower-bound $i(T)$ consists in looking for *arc-disjoint circuits*, i.e., circuits which do not share any arc (but two arc-disjoint circuits may share vertices). Indeed, if T owns q arc-disjoint circuits, it is necessary to destroy all these circuits to make T transitive. As they are arc-disjoint, reversing an arc of one of these circuits does not affect the other circuits. So, $i(T)$ is greater than or equal to the maximum number of arc-disjoint circuits of T:

Theorem 27 *Let $\nu(T)$ be the maximum number of arc-disjoint circuits of T. Then*:

$$\nu(T) \le i(T).$$

This result can also be shown by considering a maximum matching of $H(T)$. As an edge of $H(T)$ is a circuit of T, a matching of $H(T)$ defines a set of arc-disjoint circuits of T and conversely. In any hypergraph, the cardinality of any matching is lower than or equal to the cardinality of any vertex cover of the considered hypergraph (see Berge 1985). Hence, because of the identity between $\tau(T)$ and $i(T)$, the inequality (which can be strict) between $i(T)$ and the maximum cardinality $\nu(T)$ of a matching of $H(T)$.

We may also interpret $\nu(T)$ as the maximum cardinality of an independent set of the graph of which the vertices are the circuits of T and in which two vertices are linked by an edge if the associated circuits share a common arc.

But whatever the interpretation given to $\nu(T)$, the computation of this parameter remains too long in practice, even if we restrict ourselves to the search of a maximum matching in $H_3(T)$, i.e., to the search of a maximum set of 3-circuits which are arc-disjoint. Notice that, if we call $\nu_3(T)$ the cardinality of a maximum matching in $H_3(T)$, this lower bound will be less sharp than $\tau_3(T)$, since, as a consequence of the general relationship between a maximum matching and a minimum vertex cover of any hypergraph, we have the next theorem:

Theorem 28 *For any tournament T, we have $\nu_3(T) \le \tau_3(T)$.*

As a corollary of Theorem 27, it follows that $I(n)$ is lower-bounded by the maximum number $\pi(n)$ of arc-disjoint circuits that a tournament with n vertices can admit. This number is equal to the number of edge-disjoint cycles that the complete graph K_n with n vertices admits. Indeed, it is always possible, as noticed by Bermond (1972), to choose the orientations of these edge-disjoint cycles independently to transform them into arc-disjoint circuits. Then, it is sufficient to choose an arbitrary orientation for the other edges of K_n to obtain a tournament with $\pi(n)$ arc-disjoint circuits. Guy (1967) and Chartrand et al. (1971) give the value of $\pi(n)$:

Theorem 29 *For any n, we have $\pi(n) = \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor \le I(n)$.*

We may notice from the table given above that the inequality of Theorem 29 is in fact an equality for $n \le 9$, and a strict inequality for larger values of n.

We may try also to find bounds of $i(T)$ from the scores of T. The following parameter has been defined by Ryser (1964) (under the notation $\tilde{\tau}(T)$; we keep here the notation $\sigma(T)$ used in Charon et al. (1992a) in order to avoid ambiguity with the parameters τ and τ_3 used above) and has been studied by different authors, sometimes under different names (see Brualdi and Qiao 1983, 1984; Charon et al. 1992a, 1992b, 1997a; Charon and Hudry 2003, 2006; Fulkerson 1965; Monsuur and Storcken 1997; Woirgard 1997):

Definition 6

1. Let $s_1 \leq s_2 \leq \cdots \leq s_n$ be the increasing scores of T. We set:

$$\sigma(T) = \frac{1}{2}\sum_{i=1}^{n} |s_i - i + 1|.$$

2. We set $I(n,\sigma) = \mathrm{Max}\{i(T)$ for T with n vertices and verifying $\sigma(T) = \sigma\}$.

The following results are shown in Brualdi and Qiao (1983), Charon et al. (1992a) or Charon and Hudry (2003).

Theorem 30

1. *For any tournament T, $\sigma(T)$ is an integer between* 0 (*for the transitive tournaments and only for them*) *and* $(n^2 - 1)/8$ *if n is odd* (*for the regular tournaments and only for them*) *or* $(n^2 - 2n)/8$ *if n is even* (*for the quasi-regular tournaments*, *but also for some others*), *and all the intermediate integer values can be reached.*
2. *For any tournament T* (*we still assume that the scores s_i are numbered according to their increasing values*), *we have*:

$$\sigma(T) = \sum_{i:s_i - i + 1 \geq 0} (s_i - i + 1) = - \sum_{i:s_i - i + 1 \leq 0} (s_i - i + 1).$$

3. *If the vertices x_j $(1 \leq j \leq n)$ of T are numbered according to the increasing values of the scores*, *then for any q between* 1 *and* $n - 1$, *the number of arcs with their tails in* $\{x_1, x_2, \ldots, x_q\}$ *and their heads in* $\{x_{q+1}, x_{q+2}, \ldots, x_n\}$ *is less than or equal to $\sigma(T)$.*

Theorem 31 *For any tournament T, we have $\sigma(T) \leq i(T)$ and there are tournaments for which the inequality is tight.*

Theorem 32 *Let p be an integer. Then, for n large enough* ($n \geq 6p$ *suffices*):

1. *if* $p \equiv 0$ *or* 1 [mod 4], *then for* $\sigma = \frac{3p(3p+1)}{4}$, $I(n,\sigma) \geq \frac{\sqrt{16\sigma+1}-1}{6} n - \sigma$,
2. *if* $p \equiv 2$ [mod 4], *then for* $\sigma = \frac{p(9p+4)}{4}$, $I(n,\sigma) \geq \frac{\sqrt{36\sigma+4}-2}{9} n - \sigma$,
3. *if* $p \equiv 0$ *or* 1 [mod 4], *then for* $\sigma = \frac{9p^2+4p-1}{4}$, $I(n,\sigma) \geq \frac{\sqrt{36\sigma+13}-2}{9} n - \sigma$.

These lower bounds are obtained by constructing tournaments T with $\sigma(T) = \sigma$ and with a number of arc-disjoint 3-circuits equal to the lower bounds stated in Theorem 32, hence the result because of Theorem 27. Notice that these lower bounds are close to $\frac{2}{3} n\sqrt{\sigma} - \sigma$. On the other hand, for the values of σ which are not covered by the statements of Theorem 32, the construction applied to obtain these lower bounds can be adapted to obtain tournaments T with $\sigma(T) = \sigma$ and with a "large" number of arc-disjoint 3-circuits (see Charon et al. 1992a for more details).

Theorem 33 (taken from Charon and Hudry 2003) gives now an upper-bound of $I(n,\sigma)$:

Theorem 33 *For any $n \geq 1$ and any σ with* $0 \leq \sigma \leq (n^2 - 1)/8$ *if n is odd or* $0 \leq \sigma \leq (n^2 - 2n)/8$ *if n is even*, *we have*:

$$I(n,\sigma) \leq \sqrt{\left(\sigma + \frac{5}{16}\right)\left(n^2 + \sigma + \frac{5}{16}\right)} - \frac{n}{2} - \sigma.$$

The construction used in the proof of Theorem 33 to upper-bound $i(T)$ is the following. First, the vertices are gathered according to the increasing scores into about $n/(2\sqrt{\sigma})$ clusters with about $2\sqrt{\sigma}$ vertices in each cluster. Then each cluster is made transitive by reversing a number of arcs which is at most half the number of arcs inside the considered cluster. Last, all the remaining circuits are destroyed by reversing the arcs going from a cluster with low-degreed vertices towards a cluster with high-degreed vertices. As stated by Theorem 30, the number of reversed arcs from such a cluster with low-degreed vertices towards the clusters with high-degreed vertices is upper-bounded by σ. Hence the result, after several computations. Less sharp but easier upper bounds are $n(\sqrt{\sigma + \frac{5}{16}} - \frac{1}{2}) - 2(\sqrt{2} - 1)\sigma + \frac{1}{10}$ or even $n\sqrt{\sigma}$ (see Charon and Hudry 2003 for more details).

From Theorem 31 and using $n\sqrt{\sigma}$ as the upper bound, we draw the following corollary:

Corollary 7 *For any tournament T with n vertices, we have*:

$$\sigma(T) \leq i(T) \leq \lfloor n\sqrt{\sigma(T)} \rfloor.$$

Replacing $\sigma(T)$ with its maximum value does not provide an upper bound for $I(n)$ more interesting than the previous ones. Even for the small values of σ, the gap between $I(n, \sigma)$ and $n\sqrt{\sigma}$ is not negligible, as shown by Theorem 34 (see Charon et al. 1992a):

Theorem 34 *For $\sigma \in \{1, 2, 3\}$, we have*:

$$I(n, 1) = \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad I(n, 2) = \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-2}{4} \right\rfloor, \qquad I(n, 3) \in \{n-3, n-2\}.$$

Notice that the ratio between the lower bounds of $I(n, \sigma)$ stated in Theorem 32 and the upper bound stated in Theorem 33 is about $2/3$, i.e., about the same value as the ratio between the maximum number of arc-disjoint circuits and the asymptotic value of $I(n)$. As the lower bound of Theorem 32 is based on arc-disjoint circuits, this ratio seems to indicate that it should be difficult to significantly sharpen these bounds only from arc-disjoint circuits.

Still about this parameter σ, it can be interesting to compare it to other lower bounds. It is easy to design tournaments T for which $\sigma(T)$ is smaller than $\tau_3(T)$, $\chi(T)$, $\nu(T)$ or $\nu_3(T)$ (consider for instance tournaments T with $\sigma(T) = 1$). The study of the unique (see Moon 1968) regular tournament R_5 with five vertices shows that σ can also be greater than ν_3. Indeed, we have: $\sigma(R_5) = 3$ and $\nu_3(R_5) = 2$ (but R_5 owns two arc-disjoint 3-circuits and a third circuit which is arc-disjoint with respect to the previous two ones: hence $\sigma(R_5) = \nu(R_5)$). For the parameters ν and τ_3 compared to σ, we may conjecture two inequalities (Charon et al. 1997a; Woirgard 1997):

Conjecture 4

1. *For any tournament T, we have $\sigma(T) \leq \nu(T)$.*
2. *For any tournament T, we have $\sigma(T) \leq \tau_3(T)$.*

Similarly, Woirgard (1997) conjectures the following equality:

Conjecture 5 *For any tournament T, we have $\tau_3(T) = \tau(T)$.*

Another conjecture is stated in Charon et al. (1992a) about the monotonicity of $I(n,\tau)$ with τ:

Conjecture 6 *For any fixed n, $I(n,\sigma)$ increases (strictly or not?) with σ (of course, for the values of σ which are compatible with n).*

This conjecture can also be related to Bermond's ones (see above). Indeed, a large increase of $I(n,\sigma)$ would prove that, for n odd, some regular tournaments maximize Slater index (first part of Conjecture 3) and a strict increase would involve Conjecture 1 (because regular tournaments are the only ones maximizing σ for n odd). For even values of n, Conjecture 6 would not involve any Bermond's conjecture, but would reinforce the second part of Conjecture 3 by providing the structure of the scores of the tournaments (or some of them) with a maximum Slater index. The next theorem, taken from Charon and Hudry (2003), shows that the tournaments T maximizing Slater index must own a large value for their parameters $\sigma(T)$, what involves that their scores should rather be close to the ones of regular or quasi-regular tournaments.

Theorem 35 *Let T be a tournament defined on n vertices verifying $i(T)=I(n)$, with n large enough. Then*:

$$\sigma(T) > \frac{1}{8}[n^2 - 21n\sqrt{n} + 143n].$$

We may also remind the conjecture due to Adám (1964), dealing with oriented graphs (Adám's conjecture is false for general directed graphs which are not oriented, i.e., which are not antisymmetric, see Thomassen 1987).

Conjecture 7 *In any non-transitive tournament, there exists at least an arc of which the reversing makes the number of circuits decrease.*

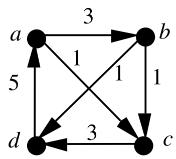
Notice that Conjecture 7 is true if "circuit" is replaced with "3-circuit" or if "number of circuits" is replaced with "Slater index". Reid (1983) shows also that Adám's conjecture is true for tournaments which are 2-arc-connected but not 3-arc-connected.

5 Bounds of the Kemeny index

As for $i(T)$, it is always possible to upper-bound the Kemeny index $K(T)$ of any tournament T with the help of a heuristic. We will not consider such a possibility here. It is also possible to upper-bound $K(T)$ by multiplying the Slater index $i(T)$ of T by the largest weight of the arcs of T or by summing the weights of the $i(T)$ heaviest weights. Unless the weights are close to each other, these bounds are usually poor, or even very poor. Obviously, the number of arcs that we must reverse in a weighted tournament T to obtain a Kemeny order cannot be less than $i(T)$, but it can be strictly larger, as shown by Fig. 5. These remarks are summarized by Lemma 2, of which the (obvious) proof is left to the reader.

Lemma 2 *Let $T=(X,U)$ be a tournament weighted by p. Let V be a subset of U of minimum cardinality such that $(X,(U-V)\cup\bar{V})$ is transitive, and let W be a subset of U with $p(W)$ minimum and such that $(X,(U-W)\cup\bar{W})$ is transitive. Then we have: $|W|\geq|V|=i(T)$ and $p(V)\geq p(W)=K(T)$.*

Fig. 5 A tournament showing that the inequalities of Lemma 2 can be strict



To show that these inequalities can be strict, consider the tournament of Fig. 5, with four vertices. The Slater index of this tournament is equal to 1: reversing the arc (d, a) suffices to make it transitive (the unique Slater order is $a > b > c > d$). But its Kemeny index is equal to 3 and corresponds to the reversing of the arcs (a, c), (b, c) and (b, d), while reversing (d, a) induces a weight equal to 5 (the unique Kemeny order is $c > d > a > b$). We may also observe on this example that it can be necessary, to obtain a Kemeny order, to reverse an arc which does not belong to any 3-circuit, as it is the case here for the arc (b, c). By the way, the tournament of Fig. 5 illustrates also the situation for which none of the Slater winners is a Kemeny winner (and conversely, of course). The links between Slater winners and Kemeny winners are also investigated in Klamler (2003) or in Charon and Hudry (2009d).

It comes also from the characterisations given above that we may deal with tournaments which represent a profile of linear orders or of tournaments, and of which all the weights are equal to 0. In this case, it is clear that any linear order is a median order and that the Kemeny index is equal to 0. It means that $K(T)$ can be equal to 0 though T is not transitive.

On the other hand, we know from Lemma 2 that $K(T)$ is upper-bounded by $i(T).p_{\max}$, where $p_{\max}$ denotes the largest weight of the arcs of T, and this bound can be reached (for instance when all the weights are equal) but, as noticed above, this bound is usually poor. The next two theorems, of which the easy proofs are left to the reader, try to sharpen these bounds.

Theorem 36 *Let T be a tournament weighted by p and let $u_1, u_2, \dots, u_{n(n-1)/2}$ be the arcs of T, numbered according to increasing weights*: $0 \le p(u_1) \le p(u_2) \le \cdots \le p(u_{n(n-1)/2})$. *Let $\alpha(T)$ (respectively $\gamma(T)$) be any lower (respectively upper) bound of $i(T)$. Then we have*:

$$\sum_{j=1}^{\alpha(T)} p(u_j) \le K(T) \le \sum_{j=0}^{\gamma(T)-1} p(u_{(n(n-1)/2)-j}).$$

This lower bound, useful for instance as the evaluation function in branch and bound methods (see below), may be applied with the parameters $\sigma(T)$ or $\chi(T)$, quite easy to compute, as $\alpha(T)$. The result is still rather poor when the weights are spread over a large range, but can be pretty good if they are close to each other. It is obviously better when $\alpha(T)$ is close to $i(T)$.

The next bound can be better than the previous one when the range of the weights is large, even if the approximation of $i(T)$ is less good than $\chi(T)$. To define it, let C be a circuit of T. Then we set $\psi(C) = \min_{u \in C} p(u)$. Similarly, for a set E of arc-disjoint circuits, we set $\Psi(E) = \sum_{C \in E} \psi(C)$. We can state Theorem 37 now:

Theorem 37 *Let E be any set of arc-disjoint circuits of T. Then we have*: $\Psi(E) \le K(T)$.

Thus we try to compute a set E of arc-disjoint circuits such that $\Psi(E)$ is as large as possible. From a practical point of view, the computation of the maximum value of $\Psi(E)$

can be too long. Anyway we can usually work with a lower bound of the maximum of $\Psi(E)$. Notice that a set E maximizing Ψ is not necessarily a set of cardinality $\nu(T)$: it can be more interesting to select less arc-disjoint circuits, but heavier. It is because of this reason (the weights of the selected circuits can be larger in average) that this lower bound can be better than the one obtained from $\chi(T)$.

6 Exact or approximation algorithms

As Kemeny's problem and Slater's problem are NP-hard, no polynomial algorithm is known to compute a Kemeny order or a Slater order (or a fortiori all the Kemeny orders or all the Slater orders), and no polynomial algorithm does exist if P and NP are distinct. Then we may adopt one of the following possibilities when we try to compute a median order of a given tournament.

- To look for one optimal solution (or all of them) with the help of an exact method with an exponential complexity. This can be done only for "small" values of n.
- To look for an approximate solution found within a "reasonable" CPU time, with the obvious aim to compute a solution as good as possible.
- To analyze the instance that we deal with in order to determine whether this instance belongs to a family of polynomial instances. If so, the studied instance can be solved by an appropriate polynomial algorithm.

Many methods have been proposed for the first two directions, based on different approaches: linear programming (by the resolution of the problem stated in Sect. 1, after having relaxed the binary constraints), dynamic programming, branch and bound methods, quadratic assignment methods, specific heuristics, metaheuristics… We give some examples below (in addition to the references given below, see also Barbut 1966; Barthélemy and Monjardet 1981, 1988; Blin and Whinston 1974, 1975; Burkard et al. 1998; Davenport and Kalagnanam 2004; Dwork et al. 2001; Flood 1990; Glover et al. 1974; Hubert and Schulz 1975; Hudry 1997a; Lawler 1964; Lempel and Cederbaum 1966; Lenstra 1977; Merchant and Rao 1976, and the references therein; see also Sect. 2 above for algorithms solving the parameterized version of the feedback arc set problem for tournaments).

6.1 Heuristics

Several specific heuristics have been suggested to rank candidates from a tournament. We detail some of them below (see also Ali et al. 1986; Baker and Hubert 1977; Becker 1967; Chenery and Watanabe 1958; Cohen et al. 1999; Cook et al. 1996; Eades and Lin 1995; Eades et al. 1993; Jacquet-Lagrèze 1969; Kano and Sakamoto 1985; Mendonça and Raghavachari 2000; Rubinstein 1980; Sakuraba and Yagiura 2009; Younger 1963 and references inside for other related methods; see also below).

Metaheuristics (local search, simulated annealing, tabu search, variable neighbourhood search, noising methods, genetic algorithms, GRASP…) have also been applied to Slater's or Kemeny's problems, on real-life or random data with n up to several hundreds. If enough time is given to these sophisticated heuristics, some of them can find optimal or at least very good solutions in a very "reasonable" time. For instance, the noising methods (see Charon and Hudry 2002 for a recent survey on these metaheuristics; noising methods can be seen as a generalization of simulated annealing and of the threshold accepting methods, see Charon and Hudry 2001a) are applied in Charon and Hudry (2006, 2009a) with a self-tuning (the user has no parameter to tune). Over the 5790 tournaments tested in Charon and Hudry (2006) with up to 100 vertices, the self-tuned noising methods found an optimal solution in all the cases but 6 (note that, as the noising methods are stochastic, a second application succeeded in solving these 6 tournaments exactly). We give below some indications on how to adapt some metaheuristics to the linear ordering problem. Other applications can be found in Belloni and Lucena (2004), Campos et al. (1999, 2001, 2005), Chaovalitwongse et al. (2009), Charon and Hudry (1998, 2009a), Chiarini et al. (2004), Congram (2000), Gamboa et al. (2006), Garcia et al. (2006), González and Pérez-Brito (2001), Greistorfer (2004), Huang and Lim (2003), Hudry (1989), Laguna et al. (1999), Schiavinotto and Stützle (2003, 2004).

Copeland scores, Borda's method and generalization to Kemeny's problem

One of the easiest method to find a "not too bad" (see also Theorem 11 above) linear order fitting a tournament consists in computing the *Copeland scores* (Copeland 1951) $s(x)$ of the candidates x. Recall (see the end of Sect. 1) that $s(x)$ is the number of candidates y such that a majority prefers x to y. From the graph theoretical point of view, $s(x)$ is the out-degree of x in the considered majority (unweighted) tournament. Of course, if the tournament is transitive, then there is only one Slater order which is given by the decreasing values of the Copeland scores (i.e., the tournament itself...). But in general, Bermond (1972) showed that ranking the candidates according to their decreasing Copeland scores (a *Copeland order*) does not always give a Slater order (though of course this may happen sometimes). This phenomenon may occur only for $n \geq 6$ (see Sect. 3). Nonetheless, this method gives orders which are not too bad. Indeed, it is proved in Coppersmith et al. (2006) that the ratio of the value found by this method and the optimal value cannot exceed 5 (see the subsection "Approximation and probabilistic algorithms" below).

This method can be extended to Kemeny's problem. One way to generalize the Copeland scores consists in considering the quantities m_x defined by:

$$m_x = \sum_{y \neq x} m_{xy}$$

(where m_{xy} still denotes the quantities defined in Sect. 1). By sorting the candidates x according to the values of the quantities m_x (with some tie-breaking rule), we obtain a linear order which generalizes the Copeland order (an improvement consists in computing a vertex x^* maximizing m_x, then in removing x^* from T, and in reiterating this process on the remaining tournament). In fact, this method is the one suggested by Borda (1784) at the end of the eighteenth century, and the quantities m_x are linked to the Borda scores $B(x)$. More precisely, we have, for any vertex x: $B(x) = (m_x + m(n-1))/2$. Of course, as for Slater's problem, this order is not necessarily an optimal one, but it is very easy and fast to compute.

Some variants more or less based on the Copeland scores have been proposed for Slater's problem, for instance, by Goddard (1983) or by Cook et al. (1988) in $O(n^4)$. They usually consist in changing the tie-breaking rule for candidates with the same score.

For instance, for Goddard's method, let $A = (a_{kl})$ be the adjacency matrix of the considered tournament. We compute the matrices (called j-connectivity matrices by Goddard) $A_j = (a^j_{kl})$ defined by $A_1 = A$ and, for any $j \geq 1$, $a^{j+1}_{kl} = (\sum_{q=1}^n a^j_{kq} + \sum_{q=1}^n a^j_{lq}) \times a_{kl}$. These computations are done until an index j appears such that all the entries a^j_{kl} are distinct. Then we adopt the linear order induced by these entries a^j_{kl}. (This method can be compared with the tournament solution based on the successive powers of A and called the "long path method" in Laslier 1997. The main differences are that, in Goddard's method, the matrices A_j are not the powers of A and that we do not consider the limit when j tends to infinity.)

Smith and Payne's method and generalizations to Kemeny's problem

As a tournament is transitive if and only if there is no 3-circuit, it is necessary and sufficient to delete all the 3-circuits of the considered tournament to make it transitive. It is shown in Moon (1968) that an arc (x, y) with $s(x) \le s(y)$, where $s(z)$ is the Copeland score of a vertex z, belongs to at least one 3-circuit. Reversing an arc to delete some 3-circuits may create new ones, but reversing an arc (x, y) with $s(x) \le s(y)$ will globally decrease the number of 3-circuits by $s(y) - s(x) + 1$. Hence the method proposed by Smith and Payne (1974). At each iteration, the arc (x, y) maximizing the quantity $s(y) - s(x) + 1$ is reversed until there is no arc (x, y) with $s(x) \le s(y)$. At the end of the algorithm, we have a linear order. This order is not necessarily an optimal solution, as shown by Phillips (1976) (a counter-example with $n = 8$ is given in Barthélemy et al. 1989 while the complexity of the algorithm is rather large (the two authors do not give the complexity of their algorithm; it can be bounded by $O(n^5)$, or less with a good data structure).

Two generalizations to Kemeny's problem have been suggested in Barthélemy et al. (1989). In the first one, an arc (x, y) maximizing the ratio $[s(y) - s(x) + 1]/m_{xy}$ is chosen. In the second, instead of the global number of 3-circuits deleted by the inversion of (x, y), we consider the weights of the 3-circuits deleted by the inversion of (x, y) minus the weights of the 3-circuits created by the inversion of (x, y). Of course these generalizations give only heuristics for Kemeny's problem.

Kaykobad et al.'s method

An improvement heuristic has been proposed by Kaykobad et al. (1995). It is initially designed for Slater's problem, but it is not difficult to extend it to weighted tournaments, with a $O(n^6)$ complexity according to the authors (or an amortized complexity in $O(n^5)$ if we accept to devote a $O(n^3)$ memory space to keep some intermediary computations). They use the notion of a *cut*. A cut in an order O is a bipartition of the vertex set X into the subset B_O of the "best" vertices according to O and the subset W_O of the "worst" vertices, still according to O. The value of the cut is the number of arcs oriented from B_O towards W_O minus the number of arcs oriented from W_O towards B_O. Noticing that an optimal order does not contain any cut with a negative value, the authors design an algorithm switching the sets B_O and W_O of the current order O when a negative cut is found.

Chanas and Kobylanski's method

Chanas and Kobylanski (1996) suggested another improvement heuristic for Kemeny's problem. It is based on the iteration of three operations called *Sort*, *Reverse* and *Insert*. When applied to an order $x_1 > \cdots > x_i > \cdots > x_n$, *Reverse* provides the reversed order $x_n > \cdots > x_i > \cdots > x_1$. Given a linear order O defined on a subset of vertices, *Insert*(x, O) adds an extra vertex x as well as possible inside O to obtain a linear order containing one more vertex and with O as its restriction to the vertices belonging to O. *Sort* is recursively defined on linear orders defined on subsets of vertices by *Sort*$(x) = x$ and, for $k > 1$, *Sort*$(x_1, \ldots, x_{k-1}, x_k) =$ *Insert*$(x_k,$ *Sort*$(x_1, \ldots, x_{k-1}))$.

Starting from a given linear order, a first stage of Chanas and Kobylanski's method consists in applying *Sort* while such an application provides a new current solution. Then a loop is applied until the current solution does not change any more. For each run of the loop, first the current solution is reversed. Then *Sort* is applied while such an application provides a new current solution. In other words, if f^* denotes the iteration of a function f while the application of f changes the current solution and if $f \circ g$ denotes the usual composition of functions (g is first applied, then f), the full method consists in applying $(((\mathit{Sort}^*) \circ \mathit{Reverse})^*) \circ (\mathit{Sort}^*)$.

Approximation and probabilistic algorithms

Probabilistic methods (see Alon and Spencer 2000 for a global presentation of these methods) have been applied to several combinatorial optimization problems for a long time, including the search of a maximum subgraph without circuit of a digraph (see Czygrinow et al. 1999; de la Vega 1983; Poljak et al. 1988; Poljak and Turzík 1986; Spencer 1971, 1978, 1987). In Poljak and Turzík (1986), a recursive algorithm is developed to solve this problem. Let $G = (V, E)$ be a graph weighted by a non-negative function w and let λ be a number with $0 < \lambda < 1$. Then, for some values of λ, the recursive algorithm finds a subgraph without circuit $H = (V, F)$ with:

$$\sum_{f \in F} w(f) \geq \lambda \sum_{e \in E} w(e) + \frac{1-\lambda}{2} \alpha(G)$$

where $\alpha(G)$ is the weight (with respect to w) of a minimum spanning tree of G. Notice that, when applied to an unweighted tournament T with $\lambda = 0.5$, this method provides a subgraph of T without circuit and with at least $(n^2 - 1)/4$ arcs. This bound is improved by the probabilistic algorithm in $O(n^3 \log n)$ designed in Poljak et al. (1988) for the same purpose: Poljak et al. show that their algorithm finds a subgraph without circuit with at least $n(n-1)/4 + n\sqrt{n}/(8\sqrt{\pi})$ arcs. By the way, this gives interesting information about the maximum number of arcs that must be reversed in a tournament to make it transitive: this number is upper-bounded by $\frac{n(n-1)}{4} - \frac{n\sqrt{n}}{8\sqrt{\pi}}$ (see Sect. 4). A generalization to weighted tournaments can be found in Czygrinow et al. (1999).

Algorithms with approximation guarantees have been studied for the feedback arc set problem (see Festa et al. 1999 for a survey on the problem for general graphs) or for the maximum acyclic subgraph problem, for general or special directed graphs (see Ausiello et al. 2003 and Vazirani 2003 for a global presentation of approximation algorithms; see Even et al. 2000; Hansen 1989; Newman 2000, 2004; Newman and Vempala 2001 or Rao and Richa 2004 for applications of these methods to some of the problems considered here). For instance, Leighton and Rao (1988) design an approximation algorithm with a $O(\log^2 n)$ approximation factor for unweighted directed graphs. This factor can be improved: Seymour (1995) shows that the feedback arc set problem can be approximated within $O(\log n \log \log n)$. On the other hand, it is known that this problem is APX-hard (Even et al. 1998) (remember that, from a practical point of view, this implies that we do not know polynomial-time approximation scheme—PTAS—to solve this problem). Similar results can be found in Berger and Shor (1990, 1997) and Hassin and Rubinstein (1994) for the maximum acyclic subgraph problem debated above, which is also APX-hard in general (Papadimitriou and Yannakakis 1991). Anyway, the maximum acyclic subgraph problem on tournaments admits a PTAS (Arora et al. 1996). Notice that an easy approximation algorithm for the maximum acyclic subgraph problem is the following (already mentioned in

Korte 1979). Place the vertices on a horizontal line and consider, on one hand, the set of the arcs oriented from the left to the right or, on the other hand, the set of the arcs oriented from the right to the left. Obviously, one of these two sets contains at least half the arcs of the tournament. As moreover an optimal solution contains at most all the arcs, it is easy to lower-bound by 0.5 the ratio between the number of arcs provided by this simple heuristic and the optimal number of arcs. Such a method can also be applied to weighted tournaments, with a same approximation factor. Anyway, this method does not provide a heuristic with an approximation guarantee for Slater's problem or, equivalently, to the feedback arc set problem applied to tournaments. This illustrates the fact that, if Problems 1 to 6 are equivalent with respect to the theory of NP-completeness, they are not necessarily such with respect to the theory of approximation.

For Slater's problem (or more precisely for the feedback arc set problem applied to unweighted tournaments), Ailon et al. (2005) design a randomized 3-approximation algorithm (another randomized algorithm, based on the linear programming formulation of the problem, can be found in Ailon et al. 2005 with an approximation ratio equal to 2.5). Their method chooses a vertex v randomly. Then, all the vertices w such that (v, w) is an arc are placed to the right of v, and the other ones to the left of v. The process runs recursively on the two sides of v. This algorithm has been derandomized by van Zuylen (2005) (see also van Zuylen and Williamson 2008), still for unweighted tournaments and with an approximation ratio equal to 3. These methods may be adapted to weighted tournaments with an approximation ratio equal to 5 (see Ailon et al. 2005 and Coppersmith et al. 2006). Alon (2006) (see also Ailon and Alon 2007), shows that, for any fixed $\varepsilon > 0$, it is NP-hard to approximate the minimum size of a feedback arc set for a tournament on n vertices up to an additive error of $n^{2-\varepsilon}$ (but approximating it up to an additive error of εn^2 can be done in polynomial time).

Interesting questions related to these considerations are the following ones:

Open problem 4

1. What is the best approximation ratio of an approximation algorithm designed to solve (approximately) the linear ordering problem?
2. Is the feedback arc set problem applied to unweighted tournaments APX-hard?

Ranking the vertices according to their scores for unweighted tournaments (Copeland's method) or to the sum of the weights of the arcs leaving them minus the sum of the weights of the arcs entering them (Borda's method) for weighted tournaments also provides a method with an approximation ratio equal to 5, irrespective of how ties are broken (see Coppersmith et al. 2006 and Fagin et al. 2005). Moreover, for any $\varepsilon > 0$, Coppersmith et al. (2006) exhibit an infinite family of unweighted tournaments such that ranking the vertices according to their scores, even with the best way to break ties, results in an order with at least $5 - \varepsilon$ times as many reversings as for a Slater order. This shows that 5 as the approximation ratio is tight for this method, even if the ties are broken in a clever way.

Metaheuristics

Many metaheuristics (among a huge literature dealing with metaheuristics, see for instance Dréo et al. 2006 or Glover and Kochenberger 2003 and the references inside) for the linear ordering problem are based on *elementary transformations* applied iteratively to a given linear order to improve it (the initial solution may be computed randomly or generated for example by one of the methods described above). An elementary transformation is a way of changing one feature of a solution without changing drastically its global structure. The "shift" operation defined above (see Definitions 1 in Sect. 3) provides such an elementary transformation (of course, other transformations can easily be designed). The *neighbourhood* $N(O)$ of an order O is the set of all the orders (the *neighbours* of O) that we can obtain by applying the elementary transformation to O.

Thanks to such an elementary transformation, we may apply metaheuristics such as an *iterative improvement method* (also called a *descent* or a *quench*). The principle of a descent is to apply the elementary transformation to the current solution O to obtain a new order $O' \in N(O)$. If this new solution is better than the current one (i.e., if we have $\Delta(O, \Pi) > \Delta(O', \Pi)$), then we adopt O' instead of O. Otherwise we keep O. Then we do it again until we reach a local optimum, i.e. an order O such that:

$$\forall O' \in N(O), \quad \Delta(O, \Pi) \leq \Delta(O', \Pi).$$

It is possible also to design more sophisticated metaheuristics based on the same idea of neighbourhood, as simulated annealing, tabu search or the noising method and so on (see references above).

Genetic algorithms (in addition to the references given above, see also Goldberg (1989) for a global presentation of these methods and for references) are not based (or rather, not mainly) on elementary transformations as defined above. These methods deal with a set of solutions (here, a set of orders) called the *individuals* of a *population*. They apply iteratively three operators usually called *selection*, *cross-over* and *mutation* to the individuals of the population. The selection phase consists in choosing some individuals of the current population in order to procreate. It is the only phase where the values of the function to minimize are taken into account. A "good" individual (i.e., an individual giving a low value to the function to minimize) is given a higher probability to be chosen as a procreator than a "bad" individual of the population. In the cross-over phase, we mix the features of two (seldom more) procreators to create one or two (seldom more) new individual(s) to constitute the next current population. The mutation phase changes randomly one (sometimes more) feature(s) of the newborn individuals in order to introduce some diversity in the population.

For our aggregation problem, genetic algorithms have been applied alone or hybridized with other metaheuristics (descents method, simulated annealing, noising methods…). For instance, we may find in Charon and Hudry (1998) the following specifications for the three genetic operators.

- Selection: the N individuals O_k $(1 \leq k \leq N)$ of the current population are sorted according to the decreasing values of $\Delta(O_k, \Pi)$. Then, if $r(O_k)$ denotes the rank of an order O_k in this sorting, the probability to choose O_k as a procreator is proportional to $r(O_k)$. It involves that the best individual of the population has a probability to be chosen equal to N times the probability of the worst one.
- Cross-over: two orders O and O' are crossed to generate a new one. Let X be the set $\{x_1, \ldots, x_n\}$. Let O be the order $x_{j_1} > x_{j_2} > \cdots > x_{j_n}$ and O' be the order $x_{k_1} > x_{k_2} > \cdots > x_{k_n}$. Each candidate x has a rank $r(x)$ in O and a rank $r'(x)$ in O'. To find the newborn child generated by O and O', we sort the candidates x according to the increasing values of the sums $r(x) + r'(x)$. The new order is given by this sorting (if several candidates have the same sum, the ties are broken randomly). For instance, if $n = 6$ with $X = \{a, b, c, d, e, f\}$ and if the two orders O and O' are $O = a > b > c > d > e > f$ and $O' = f > c > a > b > d > e$, the values of $r(x)$, $r'(x)$ and $r(x) + r'(x)$ are given in the following table. Here, the newborn child is the order $a > c > b > f > d > e$.

x	a	b	c	d	e	f
$r(x)$	1	2	3	4	5	6
$r'(x)$	3	4	2	5	6	1
$r(x)+r'(x)$	4	6	5	9	11	7

- Mutation: the mutation is given by an elementary transformation, for example a shift with randomly chosen parameters, and is applied irrespective of the variation of the objective function.

6.2 Exact algorithms

In this section, we will distinguish between two kinds of exact algorithms (other kinds exist, like dynamic programming). In a first type, we deal with methods which are exact if they succeed but which can fail in finding an order. It is the case for some methods using the linear programming form of the aggregation problem. In a second category, we find enumerative methods, of which one advantage is to be able to give all the optimal solutions. A third category, constituted by sub-problems made of polynomial families of instances, can be found in Sect. 6.3.

Methods using the linear programming formulation

While it is difficult (more precisely, NP-hard) to solve the 0-1 linear problem depicting our aggregation problem (see Sects. 1 and 2), we may more easily solve the continuous linear problem obtained by relaxing the constraints $\omega_{xy} \in \{0, 1\}$ into $0 \leq \omega_{xy} \leq 1$ (see Bowman and Colantoni 1973; de Cani 1969; Nishihara et al. 1989; Tucker 1960; Younger 1963). If, in an optimal solution of this continuous problem, all the variables ω_{xy} have integer values, then this optimal solution is also an optimal solution of the 0-1 problem. For instance, with the help of linear programming, Boenchendorf (1982) on one hand and Marcotorchino and Michaud (1979) on the other solved life problems with up to 30 vertices for Boenchendorf and up to 72 vertices for Marcotorchino and Michaud (notice that real-life problems are not always the most difficult, because of a relative consistency in the expressed preferences). Of course, the optimal solution of the relaxed problem may not be integral. In this case, we do not obtain a feasible solution for the initial problem. Another drawback of this method is that there are many variables (about n^2) and many constraints (about n^3). Then, in order to reduce the size of the bases, we may pay attention to the dual problem of the continuous relaxation. Notice also that the heuristics specified above can give a good starting solution in order to save time.

Another possibility is to relax the constraints from which the computational difficulty arises, i.e., the n^3 transitivity constraints, and then to apply Lagrangean relaxation (for a presentation of Lagrangean relaxation, see Lemaréchal 2003 for instance). This was done by Arditti (1984) and more recently in Belloni and Lucena (2004) or Charon and Hudry (2006). The same difficulty as before appears about the size of the problem: the dual (with respect to Lagrangean relaxation) problem has $O(n^3)$ variables (the Lagrangean multiplicators). In addition, the theory does not dismiss the possibility of a duality gap. In such a case, we cannot conclude and the Lagrangean relaxation provides only a bound of the optimal value of Δ. Anyway, this method allowed Arditti to solve exactly some real-life problems described in Marcotorchino and Michaud (1979) with $n = 36$ and $n = 44$ in shorter time than with the above continuous linear programming (Arditti estimates that the CPU time is

8 times less with Lagrangean relaxation than with the method applied by Marcotorchino and Michaud). The CPU time necessary in Arditti (1984) for the Lagrangean relaxation remains "reasonable": less than one hour on a DPS8 for random tournaments with n between 50 and 90.

Cutting planes and branching methods

When the previous methods fail, it is always possible to apply methods based on an exhaustive search (branch and bound; see Charon and Hudry 2009b for the principles of branch and bound methods), which may use the previous methods as components. Many attempts have been done in this direction. Of course, because of the exponential explosion, even if we limit the enumeration as much as possible, the branch and bound may require too much time and possibly too much memory space to solve a problem practically. But, on the other hand, this method can find all the optimal solutions, what the other approaches cannot do usually.

Many refinements have been included to the branch and bound methods, by using some properties verified by Slater winners or Kemeny winners (see above), by exploiting information computed during the run of the algorithm (see Charon et al. 1997a; Guénoche 1995, 1996; Guénoche et al. 1994) or by using appropriate data structure. For instance, the use of a heap to code the best-first strategy branch and bound tree in Charon et al. (1996b) reduces the complexity from $O(n!)$ in Barthélemy et al. (1989) down to $O(n \log n)$ to find the best leaf to expand. Indeed, there can be about $n!$ leaves, which may require about $n!$ steps to find the best one if they are not organized with an appropriate data structure. If they are the $n!$ elements of a heap, we may find the best leaf—the root of the heap—in $O(1)$ and, after the removal of this best leaf, we may restore the heap structure for the next search of the best leaf in $O(\log n!)$, i.e., in $O(n \log n)$. Attention has been devoted also for finding a good evaluation function for the nodes of the branch and bound tree. Cutting plane methods have also been designed and sometimes combined with polyhedral methods based on combinatorial studies of the associated polytope (see Schrijver 2003 or Wolsey 1998 for example for a global presentation of these methods) to obtain branch and cut methods in order to solve the linear ordering problem (see Barahona et al. 1994; Bolotashvili et al. 1999; Christof and Reinelt 1996, 1997a, 1997b; Christophe et al. 2004; Doignon et al. 2006; Fiorini 2001a, 2001b, 2006a, 2006b; Fiorini and Fishburn 2003; Fishburn 1992; Girlich et al. 1998; Goemans and Hall 1996; Grötschel et al. 1984a, 1984b, 1985a, 1985b, 1985c; Jünger 1985; Koppen 1995; Leung and Lee 1994; Loiseau et al. 1993; Méndez-Díaz et al. 2009; Mitchell 2000; Mitchell and Borchers 1996, 2000; Nalivaiko 1997, 1999; Nutov and Penn 1995, 1996; Reinelt 1985, 1993; Young 1978 for polyhedral approaches of the linear ordering problem or related problems). It is known that, when the problem to solve is NP-hard, we may fail in finding a complete (and irredundant) system of linear equations and inequalities to describe the solutions polytope.

Several branch and bound methods, more or less sophisticated, have been designed to solve the real-life problems quoted above (and even to enumerate all the optimal solutions) or to solve randomly generated instances with n up to 100 (see references above and for example Bermond and Kodratoff 1976; Burkov and Groppen 1972; Charon et al. 1992b, 1996b, 1997a; Charon and Hudry 2001b, 2006; Christof and Reinelt 2001; Cook and Saipe 1976; de Cani 1972; Flueck and Korsh 1974; Grötschel et al. 1984a, 1984b, 1985a, 1985b; Guénoche 1977, 1986, 1988, 1995, 1996; Guénoche et al. 1994; Hudry 1989, 1998; Jünger 1985; Kaas 1981; Korte and Oberhofer 1968, 1969; Lenstra Jr. 1973; Mitchell and Borchers 1996, 2000; Phillips 1967, 1969; Reinelt 1985; Remage and Thompson 1966; Tüshaus 1983;

Wessels 1981; Woirgard 1997; Younger 1963). For instance, Korte and Oberhofer (1968, 1969) solved random tournaments (with the same probability for the two orientations of each arc) with 13 vertices. According to Kaas (1981), Lenstra Jr. (1973) improved their software and solved problems with up to 17 vertices. Wessels (1981) increased this number to 25. The experiments of Kaas (1981) deal with tournaments with 34 vertices coming from life problems or random tournaments with 25 vertices. Thanks to their polyhedral approach and their branch and cut algorithm, Grötschel, Jünger and Reinelt solved life problems with up to 71 vertices (Christof and Reinelt 1996) and tournaments of which the weights simulate life data with up to 80 vertices (Grötschel et al. 1984a, 1984b, 1985c; Jünger 1985 and Reinelt 1985). In Charon and Hudry (2006), a branch and bound with a Lagrangean relaxation as the evaluation function is designed. The 5790 instances used to study the properties of this method include instances of Slater's problem with up to 45 vertices and instances of Kemeny's problem with up to 100 vertices (the software is freely available at the URL http://www.enst.fr/~charon/tournament/median.html; benchmarks can be found at the URL http://www.iwr.uni-heidelberg.de/groups/comopt/software/LOLIB/, Reinelt 2002; a generator of linear order instances can be found at the URL http://www.rpi.edu/~mitchj/generators/linord/, Mitchell 2007).

6.3 Special families of tournaments

About the last possibility, consisting in searching families of polynomial instances, it seems that little has been done until now. Conitzer (2006) designs a linear algorithm, based on homogeneous parts (see Sect. 3) to solve Slater's problem exactly when the result of the pairwise comparison method from which the considered tournament comes has a particular hierarchical structure.

Two algorithms in $O(n)$ (including the coding of the tournament, with an appropriate data structure) can be found in Charon et al. (1992b) to compute the Slater index and one Slater order for any tournament T with $\sigma(T) = 1$ for the first (very easy) algorithm or with $\sigma(T) = 2$ for the other (more difficult) algorithm. We can find also the following conjecture in Charon et al. (1992b):

Conjecture 8 *Let σ be a given integer. The computation of the Slater index of the tournaments T with $\sigma(T) = \sigma$ can be done in polynomial (with respect to n) time.*

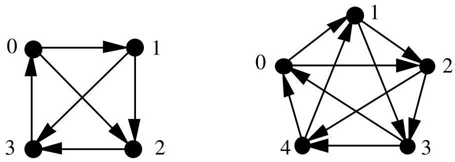
Apart from the transitive tournaments, there exists also a simple family of tournaments for which the solution of Slater's problems is easy: the so-called *cyclical* tournaments (Laslier 1997). They can be visualized as follows. The n vertices are regularly distributed on a circle and the arcs are drawn by turning around the circle, always in the same way, and beginning from the periphery. More formally, the cyclical tournaments are the ones of which the vertices can be numbered from 0 to $n - 1$ such that:

- if n is odd, there exists an arc from any vertex j $(0 \le j \le n - 1)$ towards the vertices k with $j < k \le j + (n - 1)/2$, the computations being done modulo n,
- if n is even, there exists an arc from any vertex j with $0 \le j < n/2$ towards the vertices k with $j < k \le j + n/2$ or from any vertex j with $n/2 \le j \le n - 1$ towards the vertices k with $j < k < j + n/2$, the computations still being done modulo n.

Figure 6 illustrates the cyclical tournaments on 4 or 5 vertices.

Cyclical tournaments are regular if n is odd, or quasi-regular otherwise (but there exist regular or quasi-regular tournaments which are not cyclical). Thus, from Theorem 10, if n

Fig. 6 Cyclical tournaments on 4 or 5 vertices



is odd, any vertex of a cyclical tournament T is a Slater winner of T (but this does not provide the Slater index or a Slater order of such a tournament). Theorem 38 (see Woirgard 1997) specifies the value of the Slater index and the form of the Slater orders for cyclical tournaments.

Theorem 38 *Let T be a cyclical tournament whose n vertices are numbered as described above.*

1. *If n is odd, $i(T)$ is equal to $(n^2 - 1)/8$. Moreover, T admits exactly n Slater orders of the type $j > j+1 > \cdots > j+n$ for $0 \le j \le n-1$ (the computations being done modulo n).*
2. *If n is even, $i(T)$ is equal to $(n^2 - 2n)/8$. Moreover, T admits only one Slater order: $0 > 1 > \cdots > n$.*

Thanks to Theorem 16, the result stated in Theorem 38 may be extended to the tournaments of which the strongly connected components are cyclical.

There exists another interesting family of regular tournaments: the *quadratic residues tournaments* also called *Paley tournaments*. They are defined only when n is a prime number equal to 3 modulo 4 (for example 7, 11, 19, 23, 31). In such a tournament, if the vertices are numbered from 0 to $n-1$, an arc is oriented from j towards k if $k-j$ is equal to a square modulo n. All the vertices play the same role, as well as all the pairs of vertices. As for any regular tournament, all the vertices of a Paley tournament are Slater winners. The values of their Slater indices are not known, but a good lower bound, given below and itself lowered by $\frac{n(n-1)}{4} - \frac{n\sqrt{n}}{2}(1 + \log_2 n)$, is computed in Alon and Spencer (2000).

Theorem 39 *Let n be a prime integer with $n \equiv 3$ (modulo 4) and let Q_n be the tournament of the quadratic residues on n vertices. Then we have:*

$$i(Q_n) \ge \frac{n(n-1)}{4} - \lceil \log_2 n \rceil \sqrt{n} 2^{(\lceil \log_2 n \rceil - 2)}.$$

We may observe that this lower bound is close to the bounds of $I(n)$ given by Theorem 23. So we may wonder whether these tournaments achieve the maximum values of the Slater index for the appropriate values of n:

Open problem 5 Let n be a prime integer with $n \equiv 3$ (modulo 4) and let Q_n be the tournament of the quadratic residues on n vertices. Is $i(Q_n)$ equal to $I(n)$?

Another way to exhibit families of tournaments T of which we could easily know $i(T)$ and at least one Slater order consists in starting from a set E of arcs which do not induce circuits and in building a tournament having E as its minimum feedback arc set. This is done in Barthélemy et al. (1995) (see also Isaak 1995 and Isaak and Tesman 1991). In particular, this was applied when E corresponds with directed paths, stars, disjoint arcs, complete bipartite graphs, alternated paths, alternated circuits, rooted trees, and linear orders. The Slater

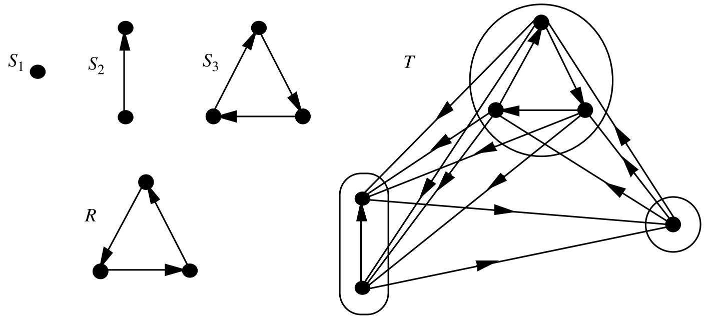


Fig. 7 An example of a substitution with $T = R \otimes [S_1, S_2, S_3]$

index of the tournaments obtained is obviously known: it is the cardinality of E. Similarly, it is obvious to obtain a Slater order of these tournaments: it suffices to reverse the arcs belonging to E. The main difficulty lies in the recognition of these tournaments. Indeed, remind that the problem of the isomorphism between two graphs is open, see Garey and Johnson (1979), and remains so for tournaments (see also Dugat 1990). From a practical point of view, this involves that checking whether two tournaments are isomorphic requires an exponential time with respect to n.

The last possibility detailed here deals with the operation called *substitution* or also *composition* (Moon 1968), *product* (Laslier 1997) or still *lexicographic product* (Bermond 1972)... It can be described as follows. We start from a tournament R on q vertices and from q tournaments S_j $(1 \le j \le q)$. Then, the jth $(1 \le j \le q)$ vertex of R is replaced with S_j. Last, for any arc (j, k) of R, all the arcs between S_j and S_k are oriented from the vertices of S_j towards the vertices of S_k. Let $R \otimes [S_1, \ldots, S_q]$ denote the resulting tournament. Figure 7 gives an example of a substitution.

The Slater index of $R \otimes [S_1, \ldots, S_q]$ can be stated in function of the Slater indices of the tournaments S_j $(1 \le j \le q)$ and of the Kemeny index of the tournament R' obtained by weighting the arcs (r_j, r_k) of R by $m_j m_k$, where m_k denotes the number of vertices of S_k. For instance, for the example of Fig. 7, the weights that should be affected to the arcs of R in order to obtain R' would be 6, 2 and 3. Theorem 40 (Woirgard 1997) specifies this relation:

Theorem 40 *With the definition of R' given above, we obtain*:

$$i(R \otimes [S_1, \ldots, S_q]) = \sum_{j=1}^{q} i(S_j) + K(R').$$

Thus, if we can recognize that a given tournament is of type $R \otimes [S_1, \ldots, S_q]$, the computation of $i(R \otimes [S_1, \ldots, S_q])$ reduces to the computations of the q Slater indices $i(S_j)$ $(1 \le j \le q)$ and of the Kemeny index $K(R')$ of the tournament R', what is usually easier because the considered tournaments are smaller. Anyway, it is not necessarily an easy task to recognize that a given tournament comes from an appropriate substitution.

When all the tournaments S_j are isomorphic to a same tournament S, we get $i(R \otimes [S, \ldots, S]) = qi(S) + m^2 i(R)$, where m denotes the number of vertices of S (Thomassen 1975), as conjectured by Bermond (1972). A special case, that we could call *self-substitution*, is the one for which S is equal to R. Then we obtain:

Corollary 8 *Let T be a tournament on n vertices. Set $T^1 = T$ and, for $k \geq 1$, $T^{k+1} = T^k \otimes [T, \ldots, T]$ where T is repeated n times inside the brackets. Let n_k ($=n^k$) denote the number of vertices of T^k. Then, for any $k \geq 1$:*

$$i(T^k) = \frac{n^{2k-1} - n^{k-1}}{n-1} i(T) = \frac{n_k^2 - n_k}{n(n-1)} i(T).$$

Notice that, if $i(T)$ is close to $n^2/4$, then $i(T^k)$ will also be close to $(n_k)^2/4$. Such a remark could be useful to study tournaments with a maximum Slater index.

7 Maximum number of Slater orders

From the voting point of view (as well as the combinatorial one!), it may be interesting to know the number of optimal solutions. As said above, the application of a branch and bound method allows to enumerate all of them. In this last section, we pay attention to the maximum number of Slater orders that an unweighted tournament with a prescribed number n of vertices may own. Notice that, for weighted tournaments, if even weights are allowed, the answer is quite simple. It is sufficient to consider the tournament with all the weights equal to 0: according to the characterization specified above, this tournament is associated with a profile of linear orders and, for this tournament, all the $n!$ linear orders are obviously optimal solutions.

It follows from Theorem 16 that the number of Slater orders may be exponential with respect to n. Indeed, consider the tournament on $n = 3q$ vertices constituted of q strongly connected components all isomorphic to a 3-circuit. As a 3-circuit owns three Slater orders, Theorem 16 shows that this tournament owns exactly $3^q = 3^{n/3} = e^{(n \ln 3)/3}$ Slater orders (notice the approximation $(\ln 3)/3 \approx 0.37$).

But we may obtain more Slater orders by considering some tournaments obtained by substitution. Let $N(T)$ denote the number of Slater orders of an unweighted tournament T or the number of median orders of a weighted tournament T. Theorem 41, taken from Woirgard (1997), gives a lower bound for the number of Slater orders that a tournament obtained by substitution can own, in function of the characteristics of the initial tournaments.

Theorem 41 *Let R be a tournament on q vertices and let $S_1, S_2, \ldots, S_q$ be q tournaments. Let R' be the weighted tournament obtained by weighting R as in Theorem 40. Then we have*:

$$N(R \otimes [S_1, \ldots, S_q]) \geq N(R') \prod_{j=1}^{q} N(S_j).$$

When the involved tournaments R and S_j $(1 \leq j \leq q)$ are all equal to a same tournament T, Theorem 41 leads to Corollary 9.

Corollary 9 *Let T be a tournament on n vertices. We set $T^1 = T$ and, for $k \geq 1$, $T^{k+1} = T^k \otimes [T, \ldots, T]$ where T is repeated n times between the brackets. Then, for any $k \geq 1$:*

$$N(T^k) \geq N(T)^{\frac{n^k-1}{n-1}}.$$

Let $N(n)$ denote the maximum number of Slater orders that a tournament on n vertices can own. By Corollary 9, we obtain a lower bound of $N(n)$, at least for some values of n (notice that there is an infinite number of such values), by choosing the initial tournament T in an appropriate manner. Thus, by choosing a 3-circuit for T, we obtain a family of tournaments T^k on 3^k vertices. Hence the inequality $N(n) \geq 3^{(n-1)/2} = \mathrm{e}^{(n \ln 3 - \ln 3)/2}$, available for any value of n which is a power of 3 (notice the approximation $(\ln 3)/2 \approx 0.55$). In fact, if the tournament R in which the substitution is done is regular (hence with an odd number of vertices) and if the substituted tournaments S_j $(1 \leq j \leq q)$ have all the same number of vertices, it is possible to prove a stronger result, as stated by Theorem 42 (Woirgard 1997).

Theorem 42 *Let R be a regular tournament on q vertices and let $S_1, S_2, \ldots, S_q$ be q tournaments with the same number n of vertices. Then*:

$$N(R \otimes [S_1, \ldots, S_q]) \geq nN(R) \prod_{j=1}^{q} N(S_j).$$

In particular, if all the tournaments S_j $(1 \leq j \leq q)$ are equal, we obtain: $N(R \otimes [S, \ldots, S]) \geq nN(R)N(S)^q$. If in addition S is also equal to R (hence $q = n$), then we obtain, with the same notation as in Corollary 9: $N(R^2) \geq nN(R)^{n+1}$ (while the number of vertices of R^2 is n^2).

By repeating $k-1$ times the substitution of 3-circuits in a 3-circuit (as done above) we obtain a tournament on $n = 3^k$ vertices with a great number of Slater orders. More precisely, we may show Corollary 10 by induction (Woirgard 1997).

Corollary 10 *Let k be a positive integer. With the same notation as above, let T be the tournament $(C_3)^k$ where C_3 is the cyclical tournament on 3 vertices. Then we have, with n denoting the number of vertices of T $(n = 3^k)$*:

$$N(T) \geq 3^{\frac{3n - 2\log_3 n - 3}{4}}.$$

Hence the inequality: $N(n) \geq \exp[\frac{\ln 3}{4}(3n - 2\log_3 n - 3)]$ when n is a power of 3 (notice the approximation $(3 \ln 3)/4 \approx 0.82$). The computation of $N(n)$ remains an open problem:

Open problem 6 Determine the maximum number $N(n)$ of Slater orders that a tournament on n vertices may have.

For any given tournament, its number of Slater orders may be lower-bounded by means of its *automorphisms group*. An automorphism can be seen as a numbering of the vertices of the tournament which keeps the orientations of the arcs. More precisely:

Definition 7 An *automorphism* φ of a tournament $T = (X, U)$ is a one-to-one correspondence from X to X such that the arc $(\varphi(x), \varphi(y))$ belongs to U if and only if (x, y) also belongs to U. The *automorphisms group* of T is the set of all the automorphisms of T (this set is a group with respect to the composition of automorphisms).

Theorem 43 (Woirgard 1997) specifies a relation between the number of automorphisms and the number of Slater orders of any tournament.

Theorem 43 *Let $G(T)$ denote the automorphisms group of T. Then $|G(T)|$ divides $N(T)$.*

As a corollary of Theorem 43, we could lower-bound $N(n)$ by the maximum $g(n)$ of the cardinality of $G(T)$ over the set of the tournaments T on n vertices. The range of $g(n)$ is given by Theorem 44 (Alspach 1968; Dixon 1967) (see also Goldberg 1966 or Moon 1968 for the exact values of $g(n)$ for $1 \le n \le 27$). Anyway, the lower bound provided by $g(n)$ in Theorem 44 when n is a power of 3 is worse than the one given by Corollary 10.

Theorem 44 *We have*: $\lim_{n\to+\infty} g(n)^{1/n} = \sqrt{3}$ *and, for any* n, $g(n) \le \sqrt{3^{n-1}}$, *with equality holding if and only if* n *is a power of* 3.

An obvious (thanks to Theorem 14) upper bound of $N(T)$ for any tournament T is given by Theorem 45:

Theorem 45 *For any tournament* T, *let* $\beta_3(T)$ *be the number of arcs which belong to some* 3-*circuits. Then we have*: $N(T) \le \binom{\beta_3(T)}{i(T)}$.

From this bound we obtain also: $N(n) \le \binom{n(n-1)/2}{I(n)}$. If we use the fact that $\binom{2k}{k}$ is asymptotically equivalent to $\frac{2^{2k}}{\sqrt{\pi k}}$ (see Matoušek and Nešetřil 1998) and since $I(n)$ is about $n(n-1)/4$, we obtain that this upper bound is not far from $\frac{2\times 2^{n(n-1)/2}}{\sqrt{\pi n(n-1)}}$ for n large enough.

To obtain a better upper bound of $N(n)$, we may exploit the fact that a Slater order of a tournament T defines a Hamiltonian path of T (see Charon and Hudry 2000). We know bounds of the maximum number $H(n)$ of Hamiltonian paths that a tournament on n vertices can own. Indeed, Alon (1990) (see also Adler et al. 2001), answering a conjecture stated by Szele (1943), proved by a probabilistic method that there exist tournaments on n vertices with at least $n!/2^{n-1}$ Hamiltonian paths and that $H(n)$ is between $n!/2^{n-1}$ and $\alpha n\sqrt{n}n!/2^n$, where α is a constant. Hence the next corollary:

Corollary 11 *There exists a constant* α *such that we have*:

$$N(n) \le \alpha n\sqrt{n}n!/2^n.$$

By the way, remember that any tournament owns an odd number of Hamiltonian paths (Rédei 1934), while the number of Slater orders may be even.

Several evidences suggest that $H(n)$ is far from $N(n)$. For instance, the comparison of $H(n)$ and $N(n)$ for the small values of n (see Poljak and Turzík 1986 for the values of $H(n)$ for $n \le 7$) shows that there is a gap even for small values of n: $H(3) = 3$, $H(4) = 5$, $H(5) = 15$, $H(6) = 45$ (and then $H(7) = 189$) while we have $N(3) = 3$, $N(4) = 3$, $N(5) = 5$, $N(6) = 9$. While it is easy, for any n, to build strongly connected tournaments with only one Slater order (like the tournaments of Fig. 8 for $n > 3$), it is known from Busch (2006) that the number of Hamiltonian paths of a strongly connected tournament is greater than or equal to $5^{(n-1)/3}$. Similarly, the study of families of some special tournaments goes towards the same conclusion. For tournaments T with $\sigma(T) \in \{1, 2\}$, the following results can be found in Hudry (1997b) or in Woirgard (1997):

Theorem 46

1. *The maximum number of Slater orders that a tournament* T *on* n *vertices and with* $\sigma(T) = 1$ *can have is equal to* $\frac{1}{3}(2^{\lfloor (n+3)/2\rfloor} + (-1)^{\lfloor (n+1)/2\rfloor})$. *Moreover this number is reached by tournaments with a maximum Slater index* (*i.e.* $\lfloor (n-1)/2\rfloor$) *over the set of tournaments* T *with* $\sigma(T) = 1$ (*see Fig.* 9).

2. *Let r_1, r_2 and r_3 be the three roots of the equation $r^3 - 5r^2 + 2r + 4 = 0$ ($r_1 \approx 4.32$; $r_2 \approx 1.27$; $r_3 \approx -0.59$). There exist three constants α, β, γ such that, for any n, there exists a tournament T on n vertices and with $\sigma(T) = 2$ for which we have: $N(T) = \alpha r_1^{\lfloor n/4 \rfloor} + \beta r_2^{\lfloor n/4 \rfloor} + \gamma r_3^{\lfloor n/4 \rfloor}$.*

On the other hand, we may prove that the maximum number of Hamiltonian paths that a tournament T with $\sigma(T) = 1$ may own is equal to $2^{n-2} + 1$, what is about the square of the maximum number of Slater orders over the same set. Moreover, the tournaments T with a maximum number of Hamiltonian paths among the tournaments T with $\sigma(T) = 1$ are of the type shown by Fig. 8 (with the same agreement as above for the missing arcs). It is easy to prove that, if n is greater than 3, such a tournament admits only one Slater order (obtained by reversing the drawn arc), which illustrates that the gap between the number of Slater orders of a tournament and the number of its Hamiltonian paths may be huge.

These differences between the shapes of the tournaments with a maximum number of Slater orders or of Hamiltonian paths lead to the following open problems:

Open problem 7

1. For a given number n of vertices, which are the tournaments with $N(n)$ Slater orders?
2. For a given number n of vertices, which are the tournaments with $H(n)$ Hamiltonian paths?
3. For a given number n of vertices, are the tournaments reaching $I(n)$ the same as the tournaments maximizing $N(n)$?

From the point of view of voting theory, it is interesting to know to which extend two optimal solutions of a given tournament may differ. Woirgard (1997) studies the maximum distance between two Slater orders of some tournaments (i.e., the diameter of the set of Slater orders of these tournaments) and the number of pairs of Slater orders which reach this maximum distance. More precisely, he showed the following result:

Theorem 47

1. *Let T be any tournament, and let O_1 and O_2 be two Slater orders of T. Then $\delta(O_1, O_2)$ is even and is between* 0 *(if and only if $O_1 = O_2$) and* $2i(T)$.
2. *There exist an infinite number of tournaments T for which the maximum distance between Slater orders of T is equal to* $2i(T)$.
3. *For any n odd, the tournaments T_1 of Fig. 9 are such that there exist $2^{i(T_1)}$ (unordered) pairs of Slater orders O_1 and O_2 of T at maximum distance: $\delta(O_1, O_2) = 2i(T_1)$, with $i(T_1) = (n-1)/2$.*

Similar results can be found in Woirgard (1997) for tournaments T with $\sigma(T) = 2$. For instance, for n of the form $4p + 3$ (see Woirgard 1997 for the other values of n), there exist tournaments T_2 on n vertices, with $\sigma(T_2) = 2$ and with a maximum Slater index over the set of tournaments T with $\sigma(T) = 2$, such that the maximum distance between any two Slater

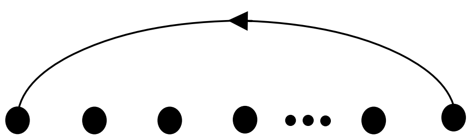


Fig. 8 Tournaments T with $\sigma(T) = 1$ and with only one Slater order and $2^{n-2} + 1$ Hamiltonian paths (the missing arcs go from the left to the right)

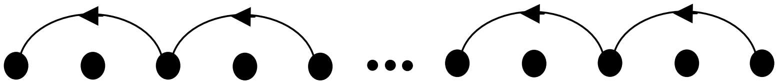


Fig. 9 Tournaments T_1 with $\sigma(T_1) = 1$, with a maximum Slater index over the set of tournaments T with $\sigma(T) = 1$, with a maximum number of Slater orders over the same set and with $2^{i(T_1)}$ pairs of Slater orders at distance $2i(T_1)$ (the missing arcs go from the left to the right)

orders O_1 and O_2 of T_2 is $5p + 2$ (respectively $5p + 3$) and with $(p + 2)2^{3p/2}$ (respectively $2^{(3p+1)/2}$) pairs of Slater orders if p is even (respectively odd). These results show that there may be many Slater orders at maximum distance. In other words, with respect to voting theory context, there may be many optimal rankings of the candidates, these optimal rankings may be quite different, and there may be many optimal rankings quite different.

This also leads to the following open problems:

Open problem 8 Let n and σ be two positive integers. Let $S_\sigma(n)$ be the set of the tournaments T defined on n vertices and verifying $\sigma(T) = \sigma$.

1. Which are the tournaments with a maximum Slater index over $S_\sigma(n)$?
2. Which are the tournaments with a maximum number of Slater orders over $S_\sigma(n)$?
3. What is the maximum distance between two Slater orders of a tournament belonging to $S_\sigma(n)$?
4. What is the maximum number of pairs of Slater orders at maximum distance for a tournament belonging to $S_\sigma(n)$?

References

Adám, A. (1964). Problem. In *Theory of graphs and its applications*. Proc. Coll. Smolenice, Czech. Acad. Sci. Publ.

Adler, I., Alon, N., & Ross, S. M. (2001). On the number of Hamiltonian paths in tournaments. *Random Structures and Algorithms*, *18*, 291–296.

Ailon, N., & Alon, N. (2007). Hardness of fully dense problems. *Information and Computation*, *205*, 117–1129.

Ailon, N., Charikar, M., & Newman, A. (2005). Aggregating inconsistent information: ranking and clustering. In *Proceedings of the 37th annual ACM symposium on theory of computing (STOC)* (pp. 684–693).

Ali, I., Cook, W. D., & Kress, M. (1986). On the minimum violations ranking of a tournament. *Management Science*, *32*, 660–674.

Alon, N. (1990). The maximum number of Hamiltonian paths in tournaments. *Combinatorica*, *10*, 319–324.

Alon, N. (2006). Ranking tournaments. *SIAM Journal on Discrete Mathematics*, *20*(1), 137–142.

Alon, N., & Spencer, J. (2000). *The probabilistic method* (2nd edn.). New York: Wiley.

Alspach, B. (1967). Cycles of each length in regular tournaments. *Canadian Mathematical Bulletin*, *10*, 283–286.

Alspach, B. (1968). A combinatorial proof of a conjecture of Goldberg and Moon. *Canadian Mathematical Bulletin*, *11*, 655–661.

Arditti, D. (1984). Un nouvel algorithme de recherche d'un ordre induit par des comparaisons par paires. In E. Diday et al. (Eds.), *Data analysis and informatics III* (pp. 323–343). Amsterdam: North-Holland.

Arora, S., Frieze, A., & Kaplan, H. (1996). A new rounding procedure for the assignment problem with applications to dense graph arrangement problems. In *Proceedings of the 37th annual IEEE symposium on foundations of computer science (FOCS)* (pp. 24–33).

Ausiello, G., Crescenzi, P., Gambosi, G., Kann, V., Marchetti-Spaccamela, A., & Protasi, M. (2003). *Complexity and approximation* (2nd edn.). Berlin: Springer.

Baker, F. B., & Hubert, L. J. (1977). Applications of combinatorial programming to data analysis: seriation using asymmetric proximity measures. *British Journal of Mathematical & Statistical Psychology*, *30*, 154–164.

Bang-Jensen, J., & Gutin, G. (2001). *Digraphs: theory, algorithms, and applications*. London, Berlin, Heidelberg: Springer.

Banks, J. (1985). Sophisticated voting outcomes and agenda control. *Social Choice and Welfare*, *2*, 295–306.

Banks, J., Bordes, G., & Le Breton, M. (1991). Covering relations, closest orderings and Hamiltonian bypaths in tournaments. *Social Choice and Welfare*, *8*, 355–363.

Bar-Noy, A., & Naor, J. (1990). Sorting, minimal feedback sets, and Hamilton paths in tournaments. *SIAM Journal on Discrete Mathematics*, *3*(1), 7–20.

Barahona, F., Fonlupt, J., & Mahjoub, A. R. (1994). Compositions of graphs and polyhedra IV: acyclic spanning subgraph. *SIDMA*, *7*(3), 390–402.

Barbut, M. (1966). Note sur les ordres totaux à distance minimum d'une relation binaire donnée. *Mathématiques et Sciences humaines*, *17*, 47–48.

Barthélemy, J.-P., & Monjardet, B. (1981). The median procedure in cluster analysis and social choice theory. *Mathematical Social Sciences*, *1*, 235–267.

Barthélemy, J.-P., & Monjardet, B. (1988). The median procedure in data analysis: new results and open problems. In H. H. Bock (Ed.), *Classification and related methods of data analysis*. Amsterdam: North-Holland.

Barthélemy, J.-P., Guénoche, A., & Hudry, O. (1989). Median linear orders: heuristics and a branch and bound algorithm. *European Journal of Operational Research*, *41*, 313–325.

Barthélemy, J.-P., Hudry, O., Isaak, G., Roberts, F. S., & Tesman, B. (1995). The reversing number of a digraph. *Discrete Applied Mathematics*, *60*, 39–76.

Barthélemy, J.-P., Cohen, G., & Lobstein, A. (1996). *Algorithmic complexity and communication problems*. London: UCL Press.

Bartholdi, J.J. III, Tovey, C.A., & Trick, M.A. (1989). Voting schemes for which it can be difficult to tell who won the election. *Social Choice and Welfare*, *6*, 157–165.

Becker, O. (1967). Das Helmstädtersche Reihenfolgeproblem—die Effizienz verschiedener Näherungsverfahren. In: *Computers uses in the social science*. Vienna.

Belloni, A., & Lucena, A. (2004). Lagrangian heuristics for the linear ordering problem. In M. G. C. Resende & J. P. de Sousa (Eds.), *Metaheuristics: computer decision making* (pp. 37–63). Dordrecht: Kluwer Academic.

Berge, C. (1985). *Graphs*. Amsterdam: North-Holland.

Berger, B., & Shor, P. W. (1997). Tight bounds for the maximum acyclic subgraph problem. *Journal of Algorithms*, *25*, 1–18.

Berger, B., & Shor, P. W. (1990). Approximation algorithms for the maximum acyclic subgraph problem. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms (SODA)* (pp. 236–243).

Bermond, J.-C. (1972). Ordres à distance minimum d'un tournoi et graphes partiels sans circuits maximaux. *Mathématiques et Sciences humaines*, *37*, 5–25.

Bermond, J.-C. (1975). The circuit-hypergraph of a tournament. In *Infinite and finite sets, proceedings of the colloquia mathematica societatis János Bolyai* (Vol. 10, pp. 165–180). Amsterdam: North-Holland,

Bermond, J.-C., & Kodratoff, Y. (1976). Une heuristique pour le calcul de l'indice de transitivité d'un tournoi. *RAIRO*, *10*(3), 83–92.

Bertacco, L., Brunetta, L., & Fischetti, M. (2008). The Linear Ordering Problem with Cumulative Costs. *European Journal of Operational Research*, *189*, 1345–1357.

Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.

Blin, J. M., & Whinston, A. B. (1974). A note on majority rule under transitivity constraints. *Management Science*, *20*, 1439–1440.

Blin, J. M., & Whinston, A. B. (1975). Discriminant functions and majority voting. *Management Science*, *21*, 1029–1041.

Boenchendorf, K. (1982). *Mathematical systems in economics: Vol. 74. Reihenfolgenprobleme/mean-flow-time sequencing*. Königstein: Verlagsgruppe Athanäum/Hain/Scriptor/Hanstein.

Bolotashvili, G., Kovalev, M., & Girlich, E. (1999). New facets of the linear ordering polytope. *SIAM Journal on Discrete Mathematics*, *12*(3), 326–336.

Borda, J.-C. (chevalier de) (1784). Mémoire sur les élections au scrutin. Histoire de l'Académie Royale des Sciences pour 1781, Paris.

Bowman, V. J., & Colantoni, C. S. (1973). Majority rule under transitivity constraints. *Management Science*, *19*, 1029–1041.

Brualdi, R. A., & Qiao, L. (1983). Upsets in round robin tournaments. *Journal of Combinatorial Theory Series B*, *35*, 62–77.

Brualdi, R. A., & Qiao, L. (1984). The interchange graph of tournaments with the same score vector. In *Progress in graph theory* (pp. 129–151). San Diego: Academic Press.

Buchheim, C., Wiegele, A., & Zheng, L. (2009). Exact algorithms for the quadratic linear ordering problem. *Informs Journal on Computing* (to appear).

Burkard, R. E., Çela, E., Pardalos, P. M., & Pitsoulis, L. S. (1998). The quadratic assignment problem. In P. M. Pardalos & D.-Z. Du (Eds.), *Handbook of combinatorial optimization* (pp. 241–338). Dordrecht: Kluwer Academic.

Burkov, V. N., & Groppen, V. O. (1972). Branch cuts in strongly connected graphs and permutation potentials. *Automation and Remote Control*, *6*, 111–119.

Busch, A. H. (2006). A note on the number of Hamiltonian paths in strong tournaments. *Electronic Journal of Combinatorics*, *13*, #N3.

Campos, V., Laguna, M., & Martí, R. (1999). Scatter search for the linear ordering problem. In D. Corne, M. Dorigo, & F. Glover (Eds.), *New ideas in optimization* (pp. 331–339). New York: McGraw-Hill.

Campos, V., Glover, F., Laguna, M., & Martí, R. (2001). An experimental evaluation of a scatter search for the linear ordering problem. *Journal of Global Optimization*, *21*(4), 397–414.

Campos, V., Laguna, M., & Martí, R. (2005). Context-independent scatter and tabu search for permutation problems. *INFORMS Journal on Computing*, *17*(1), 331–339.

Caspard, N., Monjardet, B., & Leclerc, B. (2007). *Ensembles ordonnés finis : concepts, résultats et usages*. Berlin: Springer.

Chanas, S., & Kobylanski, P. (1996). A new heuristic algorithm solving the linear ordering problem. *Computational Optimization and Applications*, *6*, 191–205.

Chaovalitwongse, W., Pardalos, P. M., Resende, M. G. C., & Grundel, D. A. (2009). Revised GRASP with path-relinking for the linear ordering problem. Submitted for publication to the *Journal of Combinatorial Optimization*, and AT&T Labs Research technical report.

Charbit, P., Thomasse, S., & Yeo, A. (2007). The minimum feedback arc set problem is NP-hard for tournaments. *Combinatorics, Probability and Computing*, *16*(1), 1–4.

Charon, I., & Hudry, O. (1998). Lamarckian genetic algorithms applied to the aggregation of preferences. *Annals of Operations Research*, *80*, 281–297.

Charon, I., & Hudry, O. (2000). Slater orders and Hamiltonian paths of tournaments. *Electronic Notes in Discrete Mathematics*, *5*, 60–63.

Charon, I., & Hudry, O. (2001a). The noising methods: a generalization of some metaheuristics. *European Journal of Operational Research*, *135*(1), 86–101.

Charon, I., & Hudry, O. (2001b). Metod vetvei i granits dlia recheniia zadatchi o lineinom poriadke na vzvechennikh tournirakh. *Discretii Analiz i Issledovanie Operatsii, Seriia 2*, *8*(2), 73–91 (in Russian).

Charon, I., & Hudry, O. (2002). The noising methods: a survey. In P. Hansen & C. C. Ribeiro (Eds.), *Essays and surveys in metaheuristics* (pp. 245–261). Dordrecht: Kluwer Academic.

Charon, I., & Hudry, O. (2003). Links between the Slater index and the Ryser index of tournaments. *Graphs and Combinatorics*, *19*(3), 309–322.

Charon, I., & Hudry, O. (2006). A branch and bound algorithm to solve the linear ordering problem for weighted tournaments. *Discrete Applied Mathematics*, *154*, 2097–2116.

Charon, I., & Hudry, O. (2007). A survey on the linear ordering problem for weighted or unweighted tournaments, *4OR*, *5*(1), 5–60.

Charon, I., & Hudry, O. (2009a). Self-tuning of the noising methods. *Optimization*, *58*(7), 1–21.

Charon, I., & Hudry, O. (2009b). Branch and bound. In V. Paschos (Ed.) *Handbook of combinatorial optimization*. Wiley, New York (to appear)

Charon, I., & Hudry, O. (2009c). Maximum distance between Slater orders and Copeland orders of tournaments (submitted for publication).

Charon, I., & Hudry, O. (2009d). How different can Slater's solutions and Kemeny's solutions of tournaments be? (submitted for publication).

Charon, I., Germa, A., & Hudry, O. (1992a). Encadrement de l'indice de Slater d'un tournoi à l'aide de ses scores. *Mathématiques, Informatique et Sciences humaines*, *118*, 53–68.

Charon, I., Germa, A., & Hudry, O. (1992b). Utilisation des scores dans des méthodes exactes déterminant les ordres médians de tournois. *Mathématiques, Informatique et Sciences humaines*, *119*, 53–74.

Charon, I., Germa, A., & Hudry, O. (1996a). Random generation of tournaments and asymmetric digraphs with given out-degrees. *European Journal of Operational Research*, *95*, 411–419.

Charon, I., Guénoche, A., Hudry, O., & Woirgard, F. (1996b). A bonsaï branch and bound method applied to voting theory. In *Ordinal and symbolic data analysis* (pp. 309–318). Berlin: Springer

Charon, I., Hudry, O., & Woirgard, F. (1996c). Ordres médians et ordres de Slater des tournois. *Mathématiques, Informatique et Sciences humaines*, *133*, 23–56.

Charon, I., Guénoche, A., Hudry, O., & Woirgard, F. (1997a). New results on the computation of median orders. *Discrete Mathematics*, *165–166*, 139–154.

Charon, I., Hudry, O., & Woirgard, F. (1997b). A 16-vertex tournament for which Banks set and Slater set are disjoint. *Discrete Applied Mathematics*, *80*(2–3), 211–215.

Chartrand, G., Geller, D., & Hedetniemi, S. (1971). Graphs with forbidden subgraphs. *Journal of Combinatorial Theory, Series B*, *10*(1), 12–41.

Chenery, H. B., & Watanabe, T. (1958). International comparisons of the structure of production. *Econometrica*, *26*(4), 487–521.

Chiarini, B., Chaovalitwongse, W., & Pardalos, P. M. (2004). A new algorithm for the triangulation of input-output tables. In P. M. Pardalos, A. Migdalas, & G. Baourakis (Eds.), *Supply chain and finance* (pp. 254–273). Singapore: World Scientific.

Christof, T., & Reinelt, G. (1996). Combinatorial optimization and small polytopes. *Top*, *4*(1), 1–64.

Christof, T., & Reinelt, G. (1997a). *Low-dimensional linear ordering polytopes*. Working paper, University of Heidelberg.

Christof, T., & Reinelt, G. (1997b). *Small instances relaxations for solving linear ordering problems by branch-and-cut* (Technical report). University of Heidelberg.

Christof, T., & Reinelt, G. (2001). Algorithmic aspects of using small instance relaxations in parallel branch-and-cut. *Algorithmica*, *30*(4), 507–629.

Christophe, J., Doignon, J.-P., & Fiorini, S. (2004). The biorder. *Order*, *21/1*, 61–82.

Cohen, W., Schapire, R., & Singer, Y. (1999). Learning to order things. *Journal of Artificial Intelligence Research*, *10*, 213–270.

Condorcet, M. J. A. N. Caritat (marquis de) (1785). Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix, Paris.

Congram, R. K. (2000). *Polynomially searchable exponential neighbourhoods for sequencing problems in combinatorial optimisation*. Ph.D. thesis, University of Southampton.

Conitzer, V. (2006). Computing Slater rankings using similarities among candidates. In *Proceedings of the 21st national conference on artificial intelligence (AAAI-06)*, Boston, MA, USA (pp. 613–619). Early version: IBM research report RC23748, 2005.

Cook, W. D., & Saipe, A. L. (1976). Committee approach to priority planning: the median ranking method. *Cahiers du Centre d'Études et de Recherche Opérationnelle 18*(3), 337–352.

Cook, W. D., Golan, I., & Kress, M. (1988). Heuristics for ranking players in a round robin tournament. *Computers and Operations Research*, *15*(2), 135–144.

Cook, W. D., Doyle, J., Green, R., & Kress, M. (1996). Ranking players in multiple tournaments. *Computers and Operations Research*, *23*(9), 869–880.

Copeland, A. H. (1951). *A "reasonable" social welfare function*. Seminar on applications of mathematics to social sciences, University of Michigan.

Coppersmith, D., & Winograd, S. (1990). Matrix multiplication via arithmetic progressions. *Journal of Symbolic Computation*, *9*, 251–280.

Coppersmith, D., Fleischer, L., & Rudra, A. (2006). Ordering by weighted number of wins gives a good ranking for weighted tournaments. In *Proceedings of the seventeenth annual ACM-SIAM symposium on discrete algorithm (SODA'06)* (pp. 776–782).

Czygrinow, A., Poljak, S., & Rödl, V. (1999). Constructive quasi-Ramsey numbers and tournament ranking. *SIAM Journal on Discrete Mathematics*, *12*(1), 48–63.

Davenport, A., & Kalagnanam, J. (2004). A computational study of the Kemeny rule for preference aggregation. In *Proceedings of the national conference on artificial intelligence (AAAI)* (pp. 697–702).

de Cani, J. S. (1969). Maximum likelihood paired comparison ranking by linear programming. *Biometrika*, *3*, 537–545.

de Cani, J. S. (1972). A branch and bound algorithm for maximum likelihood paired comparison ranking. *Biometrika*, *59*, 131–135.

de la Vega, W. F. (1983). On the maximal cardinality of a consistent set of arcs in a random tournament. *Journal of Combinatorial Theory Series B*, *35*, 328–332.

Debord, B. (1987a). Caractérisation des matrices de préférences nettes et méthodes d'agrégation associées. *Mathématiques et Sciences humaines*, *97*, 5–17.

Debord, B. (1987b). *Axiomatisation de procédures d'agrégation de préférences*. Ph.D. thesis, Université scientifique technologique et médicale de Grenoble.

Dixon, J. D. (1967). The maximum order of the group of a tournament. *Canadian Mathematical Bulletin*, *10*, 503–505.

Dodgson, C. L. (1876). *A method of taking votes on more than two issues*. Oxford: Clarendon.

Doignon, J.-P., Fiorini, S., & Joret, G. (2006). Facets of the linear ordering polytope: a unification for the fence family through weighted graphs. *Journal of Mathematical Psychology*, *50/3*, 251–262.

Dom, M., Guo, J., Hüffner, F., Niedermeier, R., & Truß, A. (2006). *Lecture notes in computer science: Vol. 3998. Fixed-parameter tractability results for feedback set problems in tournaments* (pp. 320–331). Heidelberg: Springer.

Downey, R. G., & Fellows, M. R. (1999). *Parameterized complexity*. Berlin: Springer.

Dréo, J., Petrowski, A., Taillard, E., & Siarry, P. (2006). *Metaheuristics for hard optimization. Methods and case studies*. Berlin: Springer.

Duarte, A., Laguna, M., & Marti, R. (2009). Tabu search for the linear ordering problem with cumulative costs. *Computational Optmization and Applications* (to appear).
Dugat, V. (1990). *Décomposition de tournois réguliers: théorie et application aux algorithmes de tests d'isomorphisme*. Ph.D. thesis, Université Paul Sabatier, Toulouse.
Dutta, B. (1988). Covering sets and a new Condorcet choice correspondence. *Journal of Economic Theory*, *44*, 63–80.
Dwork, C., Kumar, R., Naor, M., & Sivakumar, D. (2001). Rank aggregation methods for the Web. In *Proceedings of the 10th international conference on World Wide Web (WWW10)* (pp. 613–622).
Eades, P., & Lin, X. (1995). A new heuristic for the feedback arc set problem. *Australian Journal of Combinatorics*, *12*, 15–26.
Eades, P., Lin, X., & Smyth, W. F. (1993). A fast and effective heuristic for the feedback arc set problem. *Information Processing Letters*, *47*(6), 319–323.
Erdös, P. & Moon, J. W. (1965). On sets of consistent arcs in a tournament. *Canadian Mathematical Bulletin*, *8*, 269–271.
Erdös, P., & Moser, L. (1964). On the representation of directed graphs as unions of orderings. *Magyar Tudományos Akadémia Matematikai Kutató Intézetének Közleményei*. *9*, 125–132.
Even, G., Naor, J. S., Sudan, M., & Schieber, B. (1998). Approximating minimum feedback sets and multicuts in directed graphs. *Algorithmica*, *20*(2), 151–174.
Even, G., Naor, J. S., Rao, S., & Schieber, B. (2000). Divide-and-conquer approximation algorithms via spreading metrics. *Journal of the ACM*, *47*, 585–616.
Fagin, R., Kumar, R., Mahdian, M., Sivakumar, D., & Vee, E. (2005). Rank aggregation: an algorithmic perspective. Unpublished manuscript.
Festa, P., Pardalos, P., & Resende, M. (1999). Feedback set problems. In D. Z. Du & P. M. Pardalos (Eds.), *Handbook of combinatorial optimization* (Vol. 4, pp. 209–258). Dordrecht: Kluwer Academic.
Fiorini, S. (2001a). *Polyhedral combinatorics of order polytopes*. Ph.D. thesis, Université libre de Bruxelles.
Fiorini, S. (2001b). Determining the automorphism group of the linear ordering polytope. *Discrete Applied Mathematics*, *112/1–3*, 121–128.
Fiorini, S. (2006a). How to recycle your facets. *Discrete Optimization*, *3/2*, 136–153.
Fiorini, S. (2006b). $\{0, 1/2\}$-cuts and the linear ordering problem: surfaces that define facets. *SIAM Journal on Discrete Mathematics*, *20/4*, 893–912.
Fiorini, S., & Fishburn, P. (2003). Facets of linear signed order polytopes. *Discrete Applied Mathematics*, *131/3*, 597–610.
Fishburn, P. (1977). Condorcet social choice functions. *SIAM Journal of Applied Mathematics*, *33*, 469–489.
Fishburn, P. (1992). Induced binary probabilities and the linear ordering polytope: a status report. *Mathematical Social Sciences*, *23*, 67–80.
Flood, M. M. (1990). Exact and heuristic algorithms for the weighted feedback arc set problem: a special case of the skew-symmetric quadratic assignment problem. *Networks*, *20*, 1–23.
Flueck, J. A., & Korsh, J. F. (1974). A branch search algorithm for maximum likelihood paired comparison ranking. *Biometrika*, *61*(3), 621–626.
Fulkerson, D. R. (1965). Upsets in round robin tournaments. *Canadian Journal of Mathematics*, *17*, 957–969.
Gamboa, D., Rego, C., & Glover, F. (2006). A relax-and-cut RAMP approach for the linear ordering problem. In the abstracts booklet of ECCO XIX/CO2006 Joint Meeting, Porto, Portugal, p. 40.
Garcia, C. G., Perez-Brito, D., Campos, V., & Marti, R. (2006). Variable neighborhood search for the linear ordering problem. *Computers and Operations Research*, *33*(12), 3549–3565.
Garey, M. R., & Johnson, D. S. (1979). *Computers and intractability, a guide to the theory of NP-completeness*. New York: Freeman.
Girlich, E., Kovalev, M., & Nalivaiko, V. (1998). *A note on the extension of facet-defining digraphs* (Technical report 23/98). Faculty of Mathematics, University of Magdeburg.
Glover, F., & Kochenberger, G. A. (2003). *Handbook of metaheuristics*. Dordrecht: Kluwer Academic.
Glover, F., Klastorin, T., & Klingman, D. (1974). Optimal weighted ancestry relationships. *Management Science*, *20*, 1190–1193.
Goddard, S. T. (1983). Tournament rankings. *Management Science*, *29*(12), 1385–1392.
Goemans, M. X., & Hall, L. A. (1996). The strongest facets of the acyclic subgraph polytope are unknown. In W. H. Cunningham, S. T. McCormick, & M. Queyranne (Eds.), *Lecture notes in computer science: Vol. 1084*. *Integer programming and optimization* (pp. 415–429). Berlin: Springer.
Goldberg, M. (1966). *Results on the automorphism group of a graph*. M.Sc. Thesis, university of Alberta.
Goldberg, D. E. (1989). *Genetic algorithms in search, optimization and machine learning*. Reading: Addison-Wesley.
González, C. G., & Pérez-Brito, D. (2001). A variable neighborhood search for solving the linear ordering problem. In *Proceedings of MIC'2001-4th metaheuristics international conference* (pp. 181–185).

Greistorfer, P. (2004). Experimental pool design: input, output and combination strategies for scatter search. In M. G. C. Resende & J. P. de Sousa (Eds.), *Metaheuristics: computer decision making* (pp. 279–300). Dordrecht: Kluwer Academic.
Grindberg, E., & Dambit, Y. (1965). Some properties of graphs containing circuits. *Latvijskij Matematičeskij Ežegodnik* (pp. 65–70) (in Russian).
Grötschel, M., Jünger, M., & Reinelt, G. (1984a). A cutting plane algorithm for the linear ordering problem. *Operations Research*, *32*, 1195–1220.
Grötschel, M., Jünger, M., & Reinelt, G. (1984b). Optimal triangulation of large real-world input-output-matrices. *Statistische Hefte*, *25*, 261–295.
Grötschel, M., Jünger, M., & Reinelt, G. (1985a). On the acyclic subgraph polytope. *Mathematical Programming*, *33*, 1–27.
Grötschel, M., Jünger, M., & Reinelt, G. (1985b). Facets of the linear ordering polytope. *Mathematical Programming*, *33*, 43–60.
Grötschel, M., Jünger, M., & Reinelt, G. (1985c). Acyclic subdigraphs and linear orderings: polytopes, facets, and a cutting plane algorithm. In I. Rival (Ed.), *Graphs and orders* (pp. 217–264). Dordrecht: Reidel.
Guénoche, A. (1977). Un algorithme pour pallier l'effet Condorcet. *RAIRO*, *11*(1), 77–83.
Guénoche, A. (1986). A. B. C. D. : logiciel d'analyses booléennes et combinatoires des données, notice d'utilisation. G. R. T. C., Marseille.
Guénoche, A. (1988). Order at minimum distance of a valued tournament. Communication to *Modélisation, analyse et agrégation des préférences et des choix* (TRAP 3), Marseille-Luminy.
Guénoche, A. (1996). Vainqueurs de Kemeny et tournois difficiles. *Mathématiques, Informatique et Sciences humaines*, *133*, 57–66.
Guénoche, A., Vandeputte-Riboud, B., & Denis, J.-B. (1994). Selecting varieties using a series of trials and a combinatorial ordering method. *Agronomie*, *14*, 363–375.
Guénoche, A. (1995). How to choose according to partial evaluations. In B. Bouchon-Meunier et al. (Ed.), *Lecture notes in computer sciences: Vol. 945. Advances in intelligent computing, IPMU'94* (pp. 611–618). Berlin-Heidelberg: Springer.
Guilbaud, G. T. (1952). Les théories de l'intérêt général et le problème logique de l'agrégation. *Économie Appliquée* 5(4). Reprint in *Éléments de la théorie des jeux*, Dunod, Paris, 1968.
Guy, R. K. (1967). A coarseness conjecture of Erdös. *Journal of Combinatorial Theory*, *3*, 38–42.
Hansen, M. (1989). Approximation algorithms for geometric embeddings in the plane with applications to parallel processing problems. In *Proceedings of the 30th annual symposium on foundations of computer science* (pp. 604–609). Los Alamitos: IEEE Computer Society.
Hardouin Duparc, J. (1975). Quelques résultats sur l' « indice de transitivité » de certains tournois. *Mathématiques et Sciences humaines*, *51*, 35–41.
Hassin, R., & Rubinstein, S. (1994). Approximations for the maximum acyclic subgraph problem. *Information Processing Letters*, *51*(3), 133–140.
Hemaspaandra, L. (2000). Complexity classes. In K. H. Rosen (Ed.), *Handbook of discrete and combinatorial mathematics* (pp. 1085–1090). Boca Raton: CRC Press.
Hemaspaandra, E., Spakowski, H., & Vogel, J. (2005). The complexity of Kemeny elections. *Theoretical Computer Science*, *349*, 382–391.
Huang, G., & Lim, A. (2003). Designing a hybrid genetic algorithm for the linear ordering problem. In *LNCS: Vol. 2723. Proceedings of "genetic and evolutionary computation conference (GECCO) 2003", part I* (pp. 1053–1064). Berlin: Springer.
Huber, L. (1976). Seriation using asymmetric proximity measures. *British Journal of Mathematical & Statistical Psychology*, *29*, 32–52.
Hubert, L., & Schulz, J. (1975). Maximum likehood paired-comparison ranking and quadratic assignment. *Biometrika*, *62*(3), 655–659.
Hudry, O. (1989). *Recherche d'ordres médians : complexité, algorithmique et problèmes combinatoires*. Ph.D. thesis, ENST, Paris.
Hudry, O. (1997a). Algorithms for the aggregation of ordinal preferences: a review. In *Proceedings of the first conference on operations and quantitative management (ICOQM)*, Jaipur, India (pp. 169–176).
Hudry, O. (1997b). Nombre maximum d'ordres de Slater des tournois T vérifiant $\sigma(T) = 1$. *Mathématiques, Informatique et Sciences humaines*, *140*, 51–58.
Hudry, O. (1998). *Tournois et optimisation combinatoire*. Habilitation à diriger des recherches, université Paris 6, Paris, 1998.
Hudry, O. (2004). A note on "Banks winners in tournaments are difficult to recognize" by G. J. Woeginger. *Social Choice and Welfare*, *23*, 113–114.
Hudry, O. (2008). NP-hardness results on the aggregation of linear orders into median orders. *Annals of Operations Research*, *163*(1), 63–88.

Hudry, O. (2009a). A survey on the complexity of tournament solutions. *Mathematical Social Sciences*, *57*, 292–303.

Hudry, O. (2009b). Complexity of voting procedures. In R. Meyers (Ed.), *Encyclopedia of complexity and systems science*. New York: Springer.

Hudry, O. (2009c). Complexity of Kemeny's problems (submitted for publication).

Hudry, O. (2010). On the complexity of Slater's problems. *European Journal of Operational Research 203*, 216–221.

Hudry, O., Leclerc, B., Monjardet, B., & Barthélemy, J.-P. (2009). Metric and latticial medians. In D. Dubois, M. Pirlot, D. Bouyssou and H. Prade (Eds.), *Concepts and methods for decision aid* (pp. 763–803). Hermès (to appear).

Isaak, G. (1995). Tournaments as feedback arc sets. *The Electronic Journal of Combinatorics*, *2*, R20.

Isaak, G., & Tesman, B. (1991). The weighted reversing number of a digraph. *Congressus Numerantium*, *83*, 115–124.

Jacquet-Lagrèze É. (1969). L'agrégation des opinions individuelles. *Informatique et Sciences humaines*, *4*, 1–21.

Jung, H. A. (1970). On subgraphs without cycles in a tournament. In P. Erdös, A. Renyi, & V. T. Sös (Eds.), *Combinatorial theory and its applications II* (pp. 675–677). Amsterdam: North-Holland.

Jünger, M. (1985). *Polyhedral combinatorics and the acyclic subdigraph problem*. Berlin: Heldermann.

Kaas, R. (1981). A branch and bound algorithm for the acyclic subgraph problem. *European Journal of Operational Research*, *8*, 355–362.

Kadane, J. B. (1966). Some equivalence classes in paired comparisons. *Annals of Mathematical Statistics*, *37*, 488–494.

Kano, M., & Sakamoto, A. (1985). Ranking the vertices of a paired comparison digraph. *SIAM Journal on Algebraic and Discrete Methods*, *6*(1), 79–92.

Karp, R. (1972). Reducibility among combinatorial problems. In R. E. Miller & J. W. Tatcher (Eds.), *Complexity of computer computations* (pp. 85–103). New York: Plenum.

Kaykobad, M., Ahmed, Q. N. U., Shafiqul Khalid, A. T. M., & Bakhtiar, R.-A. (1995). A new algorithm for ranking players of a round-robin tournament. *Computers and Operations Research*, *22*(2), 221–226.

Kemeny, J. G. (1959). Mathematics without numbers. *Daedalus*, *88*, 577–591.

Kendall, M. G. (1938). *Rank correlation methods*. New York: Hafner.

Kendall, M. G., & Babington Smith, B. (1940). On the method of paired comparisons. *Biometrika*, *33*, 239–251.

Klamler, C. (2003). Kemeny's rule and Slater's rule: a binary comparison. *Economics Bulletin*, *4*(35), 1–7.

Klamler, C. (2004). The Dodgson ranking and its relation to Kemeny's method and Slater's rule. *Social Choice and Welfare*, *23*, 91–102.

Koppen, M. (1995). Random utility representation of binary choice probabilities: critical graphs yielding critical necessary conditions. *Journal of Mathematical Psychology*, *19*, 21–39.

Korte, B. (1979). Approximation algorithms for discrete optimization problems. *Annals of Discrete Mathematics*, *4*, 85–120.

Korte, B., & Oberhofer, W. (1968). Zwei Algorithmen zur Lösung eines komplexen Reihenfolgeproblems. *Unternehmensforschung*, *12*, 217–231.

Korte, B., & Oberhofer, W. (1969). Zur Triangulation von Input-Output-Matrizen. *Jahrbucher für Nationaloekonomie und Statistik*, *182*, 398–433.

Kotzig, A. (1975). On the maximal order of cyclicity of antisymmetric directed graphs. *Discrete Mathematics*, *12*, 17–25.

Lad, A., Ghani, R., Yang, Y., & Kisiel, B. (2009). Towards optimal ordering of prediction tasks. *SIAM international conference on data mining (SDM09)* (pp. 883–893).

Laffond, G., & Laslier, J.-F. (1991). Slater's winners of a tournament may not be in the Banks set. *Social Choice and Welfare*, *8*, 355–363.

Laffond, G., Laslier, J.-F., & Le Breton, M. (1993). The Bipartisan set of a tournament game. *Games and Economic Behavior*, *5*, 182–201.

Laffond, G., Laslier, J.-F., & Le Breton, M. (1991a). *Choosing from a tournament: a progress report and some new results* (Technical report). CNAM, Paris.

Laffond, G., Laslier, J.-F., & Le Breton, M. (1991b). *A game-theoretical method for ranking the participants in a tournament* (Technical report). CNAM, Paris.

Laguna, M., Marti, R., & Campos, V. (1999). Intensification and diversification with elite tabu search solutions for the linear ordering problem. *Computers and Operations Research*, *26*(12), 1217–1230.

Landau, H. G. (1953). On dominance relations and the structure of animal societies III. The condition for a score structure. *Bulletin of Mathematical Biophysics*, *13*, 1–19.

Laslier, J.-F. (1997). *Tournament solutions and majority voting*. Berlin, Heidelberg, New York: Springer.

Lawler, E. L. (1964). A comment on minimum feedback arc sets. *IEEE Transactions on Circuit Theory*, *11*, 296–297.

Leighton, T., & Rao, S. (1988). An approximation max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms. In *Proceedings of the 29th annual symposium on foundations of computer science* (pp. 422–431).

Lemaréchal, C. (2003). The omnipresence of Lagrange. *4OR*, *1*(1), 7–25.

Lempel, A., & Cederbaum, I. (1966). Minimum feedback arc and vertex sets of a directed graph. *IEEE Transactions on Circuit Theory*, *13*, 399–403.

Lenstra, J. K. (1977). *Mathematical centre tracts: Vol. 69. Sequencing by enumerative methods*. Amsterdam: Mathematisch Centrum.

Lenstra, H. W. Jr. (1973). *The acyclic subgraph problem* (Technical report BW26). Mathematisch Centrum, Amsterdam.

Leung, J., & Lee, J. (1994). More facets from fences for linear ordering and acyclic subgraph polytopes. *Discrete Applied Mathematics*, *50*, 185–200.

Loiseau, I., Mendez-Dias, I., & Nasini, G. (1993). Determinacion del rango disyuntivo de facetas del problema de ordinacion lineal. *Anales XXII JAIIO* (pp. 124–130).

Marcotorchino, J.-F., & Michaud, P. (1979). *Optimisation en analyse ordinale de données*. Paris: Masson.

Matoušek, J., & Nešetřil, J. (1998). *Invitation to discrete mathematics*. London: Clarendon Press, Oxford University Press.

McGarvey, D. (1953). A theorem on the construction of voting paradoxes. *Econometrica*, *21*, 608–610.

McKey, B. (2006). http://cs.anu.edu.au/~bdm/data/digraphs.html.

McLean, I. (1995). The first golden age of social choice, 1784–1803. In W. A. Barnett, H. Moulin, M. Salles, & N. J. Schofield (Eds.), *Social choice, welfare, and ethics: proceedings of the eighth international symposium in economic theory, econometrics* (pp. 13–33). Cambridge: Cambridge University Press.

McLean, I., & Hewitt, F. (1994). *Condorcet: foundations of social choice and political theory*. Cheltenham Glos: Edward Elgar.

McLean, I., & Urken, A. (1995). *Classics of social choice*. Ann Arbor: University of Michigan Press.

McLean, I., & Urken, A. (1997). La réception des œuvres de Condorcet sur le choix social (1794–1803) : Lhuilier, Morales et Daunou. In A.-M. Chouillet, P. Crépel (Eds.), *Condorcet, homme des lumières et de la révolution* (pp. 147–160). ENS éditions, Fontenay-aux-roses.

McLean, I., Lorrey, H., & Colomer, J. M. (2008). Social choice in medieval Europe. *Electronic Journal for History of Probability and Statistics*, 4(1).

Méndez-Díaz, I., Nasini, G., & Zabala, P. (2009). The disjunctive rank and cutting plane algorithms using of facets of the linear ordering polytope (submitted for publication).

Mendonça, D., & Raghavachari, M. (2000). Comparing the efficacy of ranking methods for multiple round-robin tournaments. *European Journal of Operational Research*, *123*, 593–605.

Merchant, D. K., & Rao, M. R. (1976). Majority decisions and transitivity: some special cases. *Management Science*, *23*(2), 125–130.

Miller, N. (1980). A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, *24*(1), 68–96.

Mitchell, J. E. (2000). Computational experience with an interior point cutting plane algorithm. *SIAM Journal on Optimization*, *10*(4), 1212–1227.

Mitchell, J. E. (2007). Generating linear ordering problems. http://www.rpi.edu/~mitchj/generators/linord/.

Mitchell, J. E., & Borchers, B. (1996). Solving real world linear ordering problems using a primal-dual interior point cutting plane method. *Annals of Operations Research*, *62*, 253–276.

Mitchell, J. E., & Borchers, B. (2000). Solving linear ordering problems with a combined interior point/simplex cutting plane algorithm. In H. L. Frenk, K. Roos, T. Terlaky, & S. Zhang (Eds.), *High performance optimization* (pp. 349–366). Dordrecht: Kluwer Academic.

Monjardet, B. (1973). Tournois et ordres médians pour une opinion. *Mathématiques et Sciences humaines*, *43*, 55–73.

Monjardet, B. (1979). Relations à éloignement minimum de relations binaires, note bibliographique. *Mathématiques et Sciences humaines*, *67*, 115–122.

Monjardet, B. (1990). Sur diverses formes de la "règle de Condorcet" d'agrégation des préférences. *Mathématiques, Informatique et Sciences humaines*, *111*, 61–71.

Monsuur, H., & Storcken, T. (1997). Measuring intransitivity. *Mathematical Social Sciences*, *34*, 125–152.

Moon, J. W. (1968). *Topics on tournaments*. New York: Holt, Rinehart and Winston.

Nalivaiko, V. (1997). The linear ordering polytope. Preprint 39/97, Faculty of Mathematics of the Otto-von-Guericke-University of Magdeburg.

Nalivaiko, V. (1999). *Das lineare Ordnungsproblem*. Ph.D. Thesis, Falkultät für Mathematik, Universität Magdeburg, Magdeburg, Germany, 1999.

Newman, A. (2000). *Approximating the maximum acyclic subgraph*. Master's thesis, MIT, USA.

Newman, A. (2004). Cuts and orderings: on semidefinite relaxations for the linear ordering problem. In K. Jansen, S. Khanna, J. D. P. Rolim, & D. Ron (Eds.), *Lectures notes in computer science* (Vol. 3122, pp. 195–206). Berlin: Springer.

Newman, A., & Vempala, S. (2001). Fences are futile: on relaxations for the linear ordering problem. In *Proceedings of the 8th conference on integer programming and combinatorial optimization (IPCO)* (pp. 333–347).

Nishihara, O., Kumamoto, H., & Inoue, K. (1989). The new formulations of minimum feedback arc set problem. In C. Brexinski (Ed.), *Numerical and applied mathematics*. J.C. Baltzer AG, Scientific Publishing Co.

Nutov, Z., & Penn, M. (1995). On the integral dicycle packings and covers and the linear ordering polytope. *Discrete Applied Mathematics*, *60*, 293–309.

Nutov, Z., & Penn, M. (1996). On non $(0, 1/2, 1)$ extreme points of the generalized transitive tournament polytope. *Linear Algebra and Its Applications*, *233*, 149–159.

Orlin, J. B. (1981). Unpublished manuscript.

Östergård, P. R. J., & Vaskelainen, V. P. (2009). A tournament of order 14 with disjoint Banks and Slater sets. *Discrete Applied Mathematics* (to appear).

Papadimitriou, C., & Yannakakis, M. (1991). Optimization, approximation, and complexity classes. *Journal of Computer System Science*, *43*, 425–440.

Phillips, J. P. N. (1967). A procedure for determining Slater's i and all nearest adjoining orders. *British Journal of Mathematical and Statistical Psychology*, *20*, 217–225.

Phillips, J. P. N. (1969). A further procedure for determining Slater's i and all nearest adjoining orders. *British Journal of Mathematical and Statistical Psychology*, *22*, 97–101.

Phillips, J. P. N. (1976). On an algorithm of Smith and Payne for determining Slater's i and all nearest adjoining orders. *British Journal of Mathematical and Statistical Psychology*, *29*, 126–127.

Poljak, S., & Turzík, D. (1986). A polynomial time heuristic for certain subgraph optimization problems with guaranteed lower bound. *Discrete Mathematics*, *58*, 99–104.

Poljak, S., Rödl, V., & Spencer, J. (1988). Tournament ranking with expected profit in polynomial time. *SIAM Journal Discrete Mathematics*, *1*(3), 372–376.

Raman, V., & Saurabh, S. (2006). Parameterized algorithms for feedback set problems and their duals in tournaments. *Theoretical Computer Science*, *351*, 446–458.

Rao, S., & Richa, A. W. (2004). New approximation techniques for some linear ordering problems. *SIAM Journal on Computing*, *34*(2), 388–404.

Ratliff, T. C. (2001). A comparison of the Dodgson method and Kemeny's rule. *Social Choice and Welfare*, *18*, 79–90.

Rédei, L. (1934). Ein kombinatorischer Satz. *Acta. Litt. Szeged*, *7*, 39–43.

Reid, K. B. (1969). On set of arcs containing no cycles in tournaments. *Canadian Mathematical Bulletin*, *12*, 261–264.

Reid, K. B. (1983). Monochromatic reachability, complementary cycles and single arc reversals in tournaments. In *Lecture notes in mathematics: Vol. 1073. Graph theory, proceedings of the first Southeast Asian graph theory colloquium held in Singapore (May 1983)* (pp. 11–21). Berlin: Springer.

Reid, K. B. (2004). Tournaments. In J. L. Gross & J. Yellen (Eds.), *Handbook of graph theory* (pp. 156–184). Boca Raton: CRC Press.

Reid, K. B., & Beineke, L. W. (1978). Tournaments. In L. W. Beineke & R. J. Wilson (Eds.), *Selected topics in graph theory* (pp. 169–204). San Diego: Academic Press.

Reinelt, G. (1985). *Research and exposition in mathematics: Vol. 8. The linear ordering problem: algorithms and applications*. Berlin: Heldermann.

Reinelt, G. (1993). A note on small linear-ordering polytopes. *Discrete & Computational Geometry*, *10*, 67–78.

Reinelt, G. (2002). Linear ordering library (LOLIB). http://www.iwr.uni-heidelberg.de/iwr/comopt/soft/LOLIB/LOLIB.html.

Remage, R., & Thompson, W. A. (1964). Rankings from paired comparison. *Annals of Mathematical Statistics*, *35*, 739–747.

Remage, R., & Thompson, W. A. (1966). Maximum likelihood paired comparison rankings. *Biometrika*, *53*, 143–149.

Righini G. (2008). A branch-and-bound algorithm for the linear ordering problem with cumulative costs. *European Journal of Operational Research*, *186*(3), 965–971.

Rubinstein, A. (1980). Ranking the participants in a tournament. *SIAM Journal of Applied Mathematics*, *98*, 108–111.

Ryser, H. J. (1964). Matrices of zeros and ones in combinatorial mathematics. In *Recent advances in matrix theory* (pp. 103–124). Madison: University of Wisconsin Press.

Sakuraba, C. S., & Yagiura, M. (2009). Efficient local search algorithms for the linear ordering problem (submitted for publication).

Schiavinotto, T., & Stützle, T. (2003). Search space analysis of the linear ordering problem. In G. R. Raidl et al. (Eds.), *Lecture notes in computer science: Vol. 2611. Applications of evolutionary computing* (pp. 322–333). Berlin: Springer.

Schiavinotto, T., & Stützle, T. (2004). The linear ordering problem: instances, search space analysis and algorithms. *Journal of Mathematical Modelling and Algorithms*, *3*(4), 367–402.

Schrijver, A. (2003). *Combinatorial optimization. Polyhedra and efficiency*. Berlin: Springer.

Schwartz, T. (1990). Cyclic tournaments and cooperative majority voting: a solution. *Social Choice and Welfare*, *7*, 19–29.

Seymour, P. (1995). Packing directed circuits fractionally. *Combinatorica*, *15*(2), 281–288.

Slater, P. (1961). Inconsistencies in a schedule of paired comparisons. *Biometrika*, *48*, 303–312.

Smith, W. D. (2007). http://rangevoting.org/PuzzDG.html.

Smith, A. F. M., & Payne, C. D. (1974). An algorithm for determining Slater's i and all nearest adjoining orders. *British Journal of Mathematical and Statistical Psychology*, *27*, 49–52.

Spencer, J. (1971). Optimal ranking of tournaments. *Networks*, *1*, 135–138.

Spencer, J. (1978). Nonconstructive methods in discrete mathematics. In G. C. Rota (Ed.), *Studies in combinatorics* (pp. 142–178). Washington: Mathematical Association of America.

Spencer, J. (1987). *CBMS-NSF regional conference series in applied mathematics: Vol. 52. Ten lectures on the probabilistic method*. Philadelphia: SIAM.

Stearns, R. (1959). The voting problem. *American Mathematical Monthly*, *66*, 761–763.

Szele, T. (1943). Kombinatorikai vizsgálatok az irányított teljes gráffal kapesolatban. *Matematikai és Fizikai Lapok*, *50*, 223–256. German translation: Untersuchungen über gerichtete vollständige Graphen. *Publicationes Mathematical Debrecen*, *13*, 145–168 (1966).

Thomassen, C. (1975). Transversals of circuits in the lexicographic product of directed graphs. *Mathématiques et Sciences humaines*, *51*, 43–45.

Thomassen, C. (1987). Counterexamples to Adám's conjecture on arc reversals in directed graphs. *Journal of Combinatorial Theory Series B*, *42*, 128–130.

Tucker, A. W. (1960). On directed graphs and integer programs. In *1960 Symposium on combinatorial problems*. Princeton: Princeton University.

Tüshaus, U. (1983). *Mathematical systems in economics: Vol. 82. Aggregation binärer Relationen in der qualitativen Datenanalyse*. Königstein: Verlagsgruppe Athenäum/Hain/Hanstein.

van Zuylen, A. (2005). *Deterministic approximation algorithms for ranking and clusterings* (Cornell ORIE Tech. Report No. 1431).

van Zuylen, A., & Williamson, D. P. (2008). Deterministic algorithms for rank aggregation and other ranking and clustering problems. In C. Kaklamanis & M. Skutella (Eds.), *LNCS: Vol. 4927. Approximation and online algorithms* (pp. 260–273). Berlin: Springer.

Vazirani, V. V. (2003). *Approximation algorithms*. Berlin: Springer.

Wakabayashi, Y. (1986). *Aggregation of binary relations: algorithmic and polyhedral investigations*. Ph.D. thesis, Augsburg university.

Wakabayashi, Y. (1998). The complexity of computing medians of relations. *Resenhas*, *3*(3), 323–349.

Wei, T. (1952). *The algebraic foundations of ranking theory*. Ph.D. thesis, Cambridge University.

Wessels, H. (1981). Triangulation und Blocktriangulation von Input-Output Tabellen. Deutsches Institut für Wirtschaftsforschung: Beiträge zur Strukturforshung, Heft 63, Berlin.

Woeginger, G. J. (2003). Banks winners in tournaments are difficult to recognize. *Social Choice and Welfare*, *20*(3), 523–528.

Woirgard, F. (1997). *Recherche et dénombrement des ordres médians des tournois*. Ph.D. thesis, ENST, Paris.

Wolsey, L. (1998). *Integer programming*. New York: Wiley.

Young, H. P. (1978). On permutations and permutation polytopes. *Mathematical Programming Study*, *8*, 128–140.

Young, H. P. (1988). Condorcet theory of voting. *American Political Science Review*, *82*, 1231–1244.

Younger, D. H. (1963). Minimum feedback arc sets for a directed graph. *IEEE Transactions on Circuit Theory*, *10*(2), 238–245.

Zermelo, E. (1929). Die Berechnung der Turnier-Ergebnisse als ein maximal Problem der Warscheinlichkeistsrechnung. *Mathematische Zeitung*, *29*, 436–460.